

Statistics of the longest interval in renewal processes

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Abstract. We consider renewal processes where events, which can for instance be the zero crossings of a stochastic process, occur at random epochs of time. The intervals of time between events, τ_1, τ_2, \dots , are independent and identically distributed (i.i.d.) random variables with a common density $\rho(\tau)$. Fixing the total observation time to t induces a global constraint on the sum of these random intervals, which accordingly become interdependent. Here we focus on the largest interval among such a sequence on the fixed time interval $(0, t)$. Depending on how the last interval is treated, we consider three different situations, indexed by $\alpha = \text{I, II and III}$. We investigate the distribution of the longest interval $\ell_{\max}^\alpha(t)$ and the probability $Q^\alpha(t)$ that the last interval is the longest one. We show that if $\rho(\tau)$ admits a well defined first moment, i.e., if it decays faster than $1/\tau^2$ for large τ , then the full statistics of $\ell_{\max}^\alpha(t)$ is given, in the large t limit, by the standard theory of extreme value statistics for i.i.d. random variables, showing in particular that the global constraint on the intervals τ_i does not play any role at large times in this case. However, if $\rho(\tau)$ exhibits heavy tails, $\rho(\tau) \sim \tau^{-1-\theta}$ for large τ , with index $0 < \theta < 1$ (like the zero-crossings of random walks corresponding to $\theta = 1/2$), we show that the fluctuations of $\ell_{\max}^\alpha(t)/t$ are governed, in the large t limit, by a stationary non-trivial universal distribution (different from a Fréchet law) which depends on both θ and α , which we compute exactly. On the other hand, $Q^\alpha(t)$ is generically different from its counterpart for i.i.d. variables (both for narrow or heavy tailed distributions $\rho(\tau)$). In particular, in the case $0 < \theta < 1$, the large t behaviour of $Q^\alpha(t)$ gives rise to universal non-trivial constants (depending also on both θ and α) which we compute exactly.

1. Introduction

Renewal processes are the simplest generalizations of the Poisson process [1, 2]. For the latter, the time intervals between successive events (e.g., the arrival of a taxi at the airport, of telephone calls, etc.) are independent and identically distributed (i.i.d.), with a common exponential distribution. In the case of simple renewal processes the time intervals between successive events are still i.i.d. but their common distribution is chosen arbitrary. This distribution can be narrow, as e.g., for a Gaussian or a uniform distribution, or broad with a heavy tail [1, 3].

The simplicity of their definition explains the ubiquity and the wide range of applications of renewal processes, both in probability theory and in statistical physics. Processes where inter-arrival times are i.i.d random variables (either exactly or as a good approximation) occur for instance in first-passage problems in Markov chains, random walks, or Brownian motion [1, 2], in the flipping of a spin for a system undergoing phase ordering [4, 5], related to some problems of occupation time [6], in blinking quantum dots [7], in persistence properties of the diffusion equation with random initial conditions [8, 9, 10], or in related questions [11, 12]. More recently, it was shown that the renewal properties which are at the heart of the theory of record statistics of random walks allow to obtain a large body of exact results for these questions [13, 14, 15, 16, 17].

In the present work we address the question of the statistics of the longest interval between successive events in renewal processes, when the process is observed between times 0 and t . Some aspects of this question were studied in the past by Lamperti [18] (see also [19]), where the situation referred to as case I in the present work was analyzed. Our aim is to perform a complete and thorough re-examination of this problem, extending previous studies in several directions. In particular we discuss the behaviour of the quantities of interest according to the nature of the distribution of intervals. In so doing, the present work marks another step forward in the systematic investigation of the properties of renewal processes, in the continuation of the study performed in [5], where a complete study of the statistics of a number of observables was presented (such as N_t , t_N , A_t , E_t defined below), as well as the statistics of the occupation time and related quantities, for either narrow distributions of intervals (such that all moments exist), or broad distributions with index $0 < \theta < 2$.

An initial study of the statistics of the longest interval between successive events in renewal processes was addressed by us in [20], where some exact results were announced and discussed in the context of stochastic processes in nonequilibrium systems. Ref. [20] was motivated, to some extent, by a previous similar question that was raised in [13] for the statistics of the longest lasting record in random walks, which involves a discrete time renewal process. More recently, the statistics of the longest lasting record has been studied in detail for symmetric random walks in [16] as well as for random walks with a drift in [15].

The questions studied in the present work naturally belong to the wide topic of extreme value statistics, which has attracted a lot of attention during these last decades, both in mathematics and in statistical physics. As is well known, the statistics of extremes for i.i.d. random variables is well understood thanks to the identification, in the limit of large samples, of three distinct universality classes: Gumbel, Fréchet and Weibull [21, 22]. However, in many physically relevant situations, the statistics of extremes shows significant deviations from the i.i.d. case and exact results are scarce. As exemplified here, renewal processes turn out to be a good laboratory to

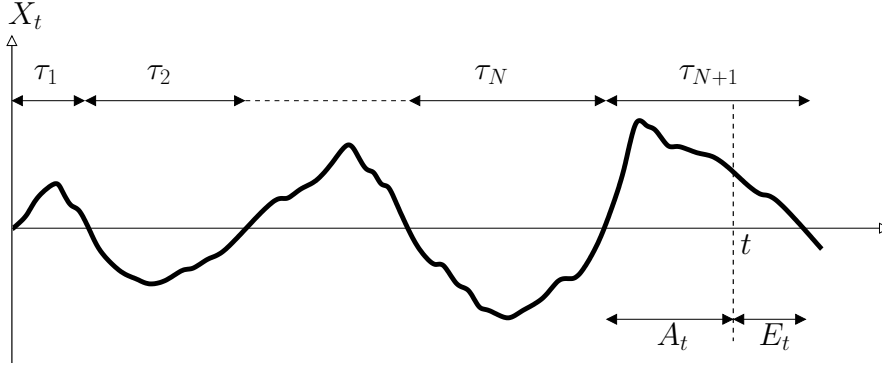


Figure 1. Illustration of a renewal process where the events are the zeros of the stochastic process X_t . The τ_i denote the intervals between the events while A_t is the backward recurrence time and E_t the forward recurrence time. For instance, in the case where X_t is Brownian motion, the distribution of the time intervals has a power-law tail, $\rho(\tau) \sim \tau^{-1-\theta}$, with $\theta = 1/2$.

test, through exact analytical results, the effects (i) of non identical random variables and (ii) of correlations between them on extreme value questions.

The statistics of the largest excursion or more generally of the largest segment of stochastic processes has been studied by several authors during the last years with applications encompassing spin-glass and disordered systems [23], the kinetics of annihilation of charged particles [24] or charged heteropolymers [25, 26]. As discussed below (see section 3), the problems that we study here, and for which we provide exact analytical results, are relevant to the questions addressed in [25, 26], related to the largest loops of random walks, which to a large extent were studied only numerically.

In the next section we give the precise definition of our model and of the quantities of interest studied in the present work. We then summarize our main results and put them in perspective with their counterparts for the case of i.i.d. random variables. The following sections are devoted to the analysis of the three cases of sequences of intervals (2.8) considered in the present study.

2. Model and results

Following [5], let us consider events occurring at the random epochs of time t_1, t_2, \dots , from some time origin $t = 0$. These events are for instance the zero crossings of some stochastic process (see figure 1). We take the origin of time on a zero crossing. This process is known as a *point process*. When the intervals of time between events, $\tau_1 = t_1, \tau_2 = t_2 - t_1, \dots$, are independent and identically distributed random variables with common density $\rho(\tau)$, the process thus formed is a *renewal process*. Hereafter we shall use indifferently the denominations: events, zero crossings or renewals.

The probability $p_0(t)$ that no event occurred up to time t , given that a zero crossing has occurred at time $t = 0$, or persistence probability [28], is simply given by the tail probability:

$$p_0(t) = \text{Prob}(\tau_1 > t) = \int_t^\infty d\tau \rho(\tau). \quad (2.1)$$

In what follows $\rho(\tau)$ will be either a narrow distribution with all moments finite, in which case the decay of $p_0(t)$, as $t \rightarrow \infty$, is faster than any power law, or a broad distribution characterized by a power-law fall-off with index θ (or persistence exponent [29, 30]):

$$p_0(t) = \int_t^\infty d\tau \rho(\tau) \approx \left(\frac{\tau_0}{t}\right)^\theta \quad (0 < \theta < 2), \quad (2.2)$$

where τ_0 is a microscopic time scale. If $\theta < 1$ all moments of $\rho(\tau)$ are divergent, while if $1 < \theta < 2$, the first moment $\langle \tau \rangle$ is finite but higher moments are divergent. In Laplace space, where s is conjugate to τ , for a narrow distribution we have

$$\mathcal{L}_\tau \rho(\tau) = \hat{\rho}(s) = \int_0^\infty d\tau e^{-s\tau} \rho(\tau) \underset{s \rightarrow 0}{=} 1 - \langle \tau \rangle s + \frac{1}{2} \langle \tau^2 \rangle s^2 + \dots \quad (2.3)$$

The above notations for the Laplace transform will be used throughout the paper. For a broad distribution, (2.2) yields

$$\hat{\rho}(s) \underset{s \rightarrow 0}{\approx} \begin{cases} 1 - a s^\theta & (\theta < 1) \\ 1 - \langle \tau \rangle s + a s^\theta & (1 < \theta < 2), \end{cases} \quad (2.4)$$

with $a = |\Gamma(1 - \theta)|\tau_0^\theta$.

Let us recall the definitions of a few natural and fundamental quantities associated to a renewal process [1, 2, 5]. Firstly, we denote by N_t the number of events which occurred between 0 and t , i.e., the largest n such that $t_n \leq t$. The time of occurrence of the last event before t , that is of the N_t -th event, is therefore the sum of a random number of random variables[‡]

$$t_N = \tau_1 + \dots + \tau_{N_t}. \quad (2.5)$$

The backward recurrence time A_t (see figure 1) is defined as the length of time measured backwards from t to the last event before t , i.e.,

$$A_t = t - t_N. \quad (2.6)$$

It is therefore the age of the current, unfinished, interval at time t . Finally the forward recurrence time (also called the excess time or residual time) E_t is the time interval between t and the next event (see figure 1),

$$E_t = t_{N+1} - t. \quad (2.7)$$

We have the simple relation $A_t + E_t = t_{N+1} - t_N = \tau_{N+1}$.

In this work, our focus is on the longest interval between two events. When considering such a renewal process on a fixed time interval $(0, t)$, the last interval plays a singular role (see figure 1). Hence, following [16, 20], it is natural to distinguish three possible sequences of intervals (or configurations) of interest,

$$\begin{aligned} \mathcal{C}^I &= \{\tau_1, \tau_2, \dots, \tau_N, A_t\}, \\ \mathcal{C}^{II} &= \{\tau_1, \tau_2, \dots, \tau_N, \tau_{N+1}\}, \\ \mathcal{C}^{III} &= \{\tau_1, \tau_2, \dots, \tau_N\}, \end{aligned} \quad (2.8)$$

which accordingly yield three cases for the longest interval,

$$\begin{aligned} \ell_{\max}^I(t) &= \max(\tau_1, \tau_2, \dots, \tau_N, A_t), \\ \ell_{\max}^{II}(t) &= \max(\tau_1, \tau_2, \dots, \tau_N, \tau_{N+1}), \\ \ell_{\max}^{III}(t) &= \max(\tau_1, \tau_2, \dots, \tau_N), \end{aligned} \quad (2.9)$$

[‡] When no ambiguity arises, we drop the time dependence of the random variable if the latter is itself in subscript.

as well as for the probability that this longest interval be the last one, or probability of record breaking:

$$\begin{aligned} Q^{\text{I}}(t) &= \text{Prob}(A_t > \max(\tau_1, \tau_2, \dots, \tau_N)), \\ Q^{\text{II}}(t) &= \text{Prob}(\tau_{N+1} > \max(\tau_1, \tau_2, \dots, \tau_N)), \\ Q^{\text{III}}(t) &= \text{Prob}(\tau_N > \max(\tau_1, \tau_2, \dots, \tau_{N-1})). \end{aligned} \quad (2.10)$$

As mentioned above, the study of the quantities $\ell_{\max}^\alpha(t)$ and $Q^\alpha(t)$, where $\alpha = \text{I, II or III}$, have been the subject of several recent studies, which we briefly summarize. Refs. [13, 15] were concerned with the study of the average value $\langle \ell_{\max}^{\text{I}}(t) \rangle$. In [20], we focused on $\langle \ell_{\max}^\alpha(t) \rangle$ (when it exists) for all three cases $\alpha = \text{I, II, III}$, as well as on $Q^{\text{I}}(t)$ (thus restricting to the case $\alpha = \text{I}$ for this quantity), for power-law distributions of intervals with index $0 < \theta < 2$. More recently, in [16], we studied the quantities $\ell_{\max}^\alpha(t)$ and $Q^\alpha(t)$, for all α , in the context of record statistics of random walks. The associated renewal process is thus defined in discrete time with tail exponent $\theta = 1/2$ (as for the excursions of Brownian motion).

Here, we compute the full distribution of $\ell_{\max}^\alpha(t)$ for any distribution of intervals $\rho(\tau)$ and for the three cases $\alpha = \text{I, II, III}$. The rest of this section is devoted to a brief presentation of our main results.

Though the random variables τ_1, τ_2, \dots , drawn from the common distribution $\rho(\tau)$, are a priori independent and identically distributed, the induced random variables $\tau_1, \dots, \tau_N, \tau_{N+1}, A_t$ occurring in the sequences \mathcal{C}^α are neither all identically distributed (except for the N_t first ones, as in case III) nor independent, because fixing an observation time t implies the interdependence of these intervals[§]. In particular, any of the first N_t intervals in \mathcal{C}^α is obviously smaller than t . While, in the first situation, fixing the number n of intervals implies that the sum

$$t_n = \tau_1 + \dots + \tau_n \quad (2.11)$$

fluctuates, in the second situation, t is fixed but the number of variables, N_t , fluctuates. This is reminiscent of what occurs when changing ensembles in statistical mechanics. We are thus naturally led to put our results in perspective with their counterparts for the case of n i.i.d. random intervals τ_i .

Let us recall that, if

$$\tau_{\max}(n) = \max(\tau_1, \dots, \tau_n) \quad (2.12)$$

is the maximum of the n i.i.d. positive random variables τ_1, \dots, τ_n , then, according to the nature of the distribution $\rho(\tau)$ of these variables, the asymptotic distribution of this maximum, after appropriate rescaling, falls in one of three classes, namely, the Gumbel class (narrow distribution with unbounded support), the Weibull class (narrow distribution with bounded support) and the Fréchet class (broad distribution with a power-law tail of index $\theta > 0$) [21]. It is also well known that, for these i.i.d. variables, the probability of record breaking is given by [31]

$$Q(n) = \text{Prob}(\tau_n > \max(\tau_1, \dots, \tau_{n-1})) = \frac{1}{n}, \quad (2.13)$$

irrespectively of the distribution of these random variables.

[§] A discussion of this point is given in section 5 of [5]. In (2.11) the random variables are independent, in (2.5) they are not.

Let us first discuss the statistics of $\ell_{\max}^\alpha(t)$. For a narrow distribution of intervals, with finite moments, one has [1, 2, 5]

$$\langle N_t \rangle \approx \frac{t}{\langle \tau \rangle}, \quad (2.14)$$

One thus expects asymptotic equivalence between the statistics of $\ell_{\max}^\alpha(t)$ and that of its counterpart for i.i.d. random variables, $\tau_{\max}(n)$. This is corroborated by our results. As shown in table 1, for the simplest renewal process with exponential distribution of intervals, $\rho(\tau) = e^{-\tau}$, the logarithmic growth of $\langle \ell_{\max}^\alpha(t) \rangle \approx \ln t + \gamma$, where $\gamma = 0.577216\dots$ is the Euler constant, is similar to the one obtained for $n \sim t$ i.i.d. random variables, $\langle \tau_{\max}(n) \rangle \approx \ln n + \gamma$. In addition, we also show that the distribution of $(\ell_{\max}^\alpha(t) - \ln t)$ is given, in the large t limit, by a Gumbel distribution (3.23), as for i.i.d. random variables. This correspondence can be generalized to any narrow distribution of intervals with unbounded support. For instance, for a Gaussian distribution, we have now $\langle \ell_{\max}^\alpha(t) \rangle \approx (\ln t / \langle \tau \rangle)^{1/2}$, which is in line with its counterpart for i.i.d. random variables where $\langle \tau_{\max}(n) \rangle \approx (\ln n)^{1/2}$, and the properly rescaled variable asymptotically follows the Gumbel law. Likewise, for a narrow distribution with a bounded support, such as, for instance, a uniform distribution of intervals, we have $\langle \ell_{\max}^\alpha(t) \rangle \approx 1 - \langle \tau \rangle / t$, which is in line with its counterpart for i.i.d. random variables where $\langle \tau_{\max}(n) \rangle \approx 1 - 1/n$, and the rescaled maximum is exponential (which is a special case of a Weibull distribution).

For a distribution with a power-law distribution of intervals with index $\theta > 1$, (2.14) still holds, hence, again, the statistics of $\ell_{\max}^\alpha(t)$ and $\tau_{\max}(n)$ are expected to be asymptotically equivalent. This is indeed the case: table 2 gives the scaling $\langle \ell_{\max}^\alpha(t) \rangle \sim t^{1/\theta}$, and as shown in the text, the rescaled variable $\ell_{\max}^\alpha(t)/t^{1/\theta}$ asymptotically converges to a Fréchet random variable Z^F (3.59) of index θ , for the three cases $\alpha = \text{I, II, III}$. Hence this shows that if the distribution $\rho(\tau)$ has a well defined first moment, i.e., if $\rho(\tau)$ decays faster than $1/\tau^2$ for large τ (this includes the case where $\rho(\tau)$ vanishes identically beyond a certain value τ_M), the limiting distribution of $\ell_{\max}(t)$, properly shifted and scaled, is the same as for i.i.d. random variables. This implies that neither the global constraint on the sum of the intervals τ_i nor the fact that the sequences \mathcal{C}^α (2.8) consists of non-identical variables (for the cases $\alpha = \text{I}$ and $\alpha = \text{II}$) plays a role in the statistics of $\ell_{\max}(t)$ for large t .

α	$\langle \ell_{\max}^\alpha(t) \rangle$	$Q^\alpha(t)$
I	$\approx \ln t + \gamma$	$\approx 1/t$
II	$\approx \ln t + \gamma$	$\approx \ln t / t$
III	$\approx \ln t + \gamma$	$\approx 1/t$

Table 1: Asymptotic results at large times for an exponential distribution of intervals $\rho(\tau)$, where γ is the Euler constant.

The situation is quite different, and more interesting from the point of view of the statistics of extremes, in the case where the distribution of time intervals has a power-law tail $\rho(\tau) \sim \tau^{-1-\theta}$ with index $0 < \theta < 1$.

Let us first present a heuristic argument yielding the typical behaviour of $\ell_{\max}(t)$ as a function of time. For i.i.d. random variables it was pointed out by Lévy [32] that the largest term of the sum outshadows the contribution of all the other terms. This

α	$\langle \ell_{\max}^\alpha(t) \rangle$	$Q^\alpha(t)$
I	$\sim t^{1/\theta}$	$\sim t^{1/\theta-1}$
II	$\sim t^{1/\theta}$	$\sim t^{1/\theta-1}$
III	$\sim t^{1/\theta}$	$\approx \langle \tau \rangle / t$

Table 2: Asymptotic results at large times for a broad distribution of intervals $\rho(\tau)$ with $\theta > 1$.

statement can be made more precise as follows. The sum (2.11) of the n i.i.d. positive random variables τ_1, \dots, τ_n scales as

$$t_n \sim n^{1/\theta} X(\theta, 1), \quad (2.15)$$

denoting by $X(\theta, 1)$ the one-sided stable law of index θ (and asymmetry parameter equal to 1). On the other hand,

$$\tau_{\max}(n) \sim n^{1/\theta} Z^F \quad (2.16)$$

where Z^F is a random variable. Hence, the rescaled variable $t_n/\tau_{\max}(n)$ is expected to have a limiting distribution, denoted by f_W , corresponding to the random variable $W = X(\theta, 1)/Z^F$ (where $X(\theta, 1)$ and Z^F are not independent). This is actually the case, as shown by Darling [33]. The characteristic function of W , given in [33] (theorem 5.1), is easily translated in Laplace space [1] (see (3.46) below).

Let us now perform a parallel reasoning for our study, arguing as follows. The scaling between N_t and t reads:

$$N_t \sim t^\theta Y_t, \quad (2.17)$$

where Y_t has a limiting distribution, $Y_t \rightarrow X(\theta, 1)^{-\theta}$, where $X(\theta, 1)$ is the one-sided stable law of index θ mentioned above (see [5] for a simple proof). Using (2.17) in order to translate n into t , and $\tau_{\max}(n)$ into $\ell_{\max}(t)$, we thus infer the scaling behaviour

$$\ell_{\max}(t) \sim t Y_t^{1/\theta} Z^F \sim t \frac{Z^F}{X(\theta, 1)}, \quad (2.18)$$

implying the existence of the same limiting distribution for the rescaled variable $t/\ell_{\max}(t)$ as for $t_n/\tau_{\max}(n)$. Though it is difficult to predict from this heuristic reasoning whether the quantity $\ell_{\max}(t)$ thus defined can be identified to one of the observables $\ell_{\max}^\alpha(t)$, if any, it nevertheless gives an argument in favour of the existence of limiting distributions for the ratios $t/\ell_{\max}^\alpha(t)$.

Turning to our results, the predicted scaling (2.18) is corroborated by table 3. Furthermore, the rescaled variables $V_t^\alpha = t/\ell_{\max}^\alpha(t)$ have indeed limiting distributions, given in Laplace space respectively by (3.29) for case I, (4.17) for case II and (5.19) for case III, and depicted in figure 2. Figures 4 (case I) and 8 (case III) depict these distributions in real space, while figures 3, 6, 7 depicts the distributions of their inverses (all these figures for $\theta = 1/2$). A striking fact is that *the distribution of Darling, f_W , for i.i.d. variables, is identical to our prediction for f_V^I* , given in Laplace space by (3.29). There is thus asymptotic equivalence between $t/\ell_{\max}^I(t)$ for case I and $t_n/\tau_{\max}(n)$ for the case of i.i.d random intervals. The distribution of $t/\ell_{\max}^I(t)$ was previously computed by Lamperti [18], and its connection with the distribution of Darling is pointed out in [34]. We provide here a simple alternative method, different from the one of [18], to compute the distribution of $t/\ell_{\max}^I(t)$.

α	$\langle \ell_{\max}^\alpha(t) \rangle / t$	$Q^\alpha(t)$
I	$\approx 0.6265 \dots$	$\approx 0.6265 \dots$
II	∞	$\approx 0.8001 \dots$
III	$\approx 0.2417 \dots$	$\approx (A \ln t + B)/t^{1/2}$

Table 3: Asymptotic results at large times for a broad distribution of intervals $\rho(\tau)$ with $\theta = 1/2$. The constants A and B are respectively equal to $1/2$ and $\gamma/2 + \ln(2/\pi) \approx 0.4094 \dots$

Let us close by discussing our results concerning the probability of record breaking $Q^\alpha(t)$. Tables 2 and 3 show that, in general, this probability is different from its counterpart $Q(n)$ for i.i.d. variables (2.13). This difference can be seen for cases I and II, except in the special case of an exponential distribution, as shown in table 1 where there is asymptotic equivalence with the i.i.d. situation, for case I, but not even for case II. For any other narrow distribution this equivalence is lost. In case III, τ_1, \dots, τ_N are exchangeable, hence $Q^{\text{III}}(t) = \langle 1/N_t \rangle$, which translates into asymptotic equivalence with $Q(n)$, not complete yet, since, as shown in table 3, there is a logarithm in the numerator of $Q^{\text{III}}(t)$.

Let us note that for $0 < \theta < 1$, the asymptotic values of $Q^\alpha(t)$ are characterized by non-trivial universal constants for cases I and II, but not for case III (see table 3).

3. Case I

3.1. Distribution of $\ell_{\max}^{\text{I}}(t)$

We first expound the general method, then discuss the results according to the nature of the distribution $\rho(\tau)$ of intervals.

We begin by determining the distribution function of

$$\ell_{\max}^{\text{I}}(t) = \max(\tau_1, \dots, \tau_N, A_t), \quad (3.1)$$

denoted by

$$F^{\text{I}}(t; \ell) = \text{Prob}(\ell_{\max}^{\text{I}}(t) \leq \ell). \quad (3.2)$$

This function bears an explicit dependence in time t , which plays a fundamental role in the definition of the renewal process and in the discussion hereafter, besides its dependence in the temporal variable ℓ associated to $\ell_{\max}^{\text{I}}(t)$. We consider the joint probability distribution of $\ell_{\max}^{\text{I}}(t)$ and N_t ,

$$\begin{aligned} F_n^{\text{I}}(t; \ell) &= \text{Prob}(\ell_{\max}^{\text{I}}(t) \leq \ell, N_t = n) \\ &= \int_0^\ell d\ell_1 \dots \int_0^\ell d\ell_n \int_0^\ell da f^{\text{I}}(t; \ell_1, \dots, \ell_n, a) \\ &= \int_0^\ell d\ell_1 \rho(\ell_1) \dots \int_0^\ell d\ell_n \rho(\ell_n) \int_0^\ell da p_0(a) \delta\left(\sum_{i=1}^n \ell_i + a - t\right), \end{aligned} \quad (3.3)$$

where $\{\ell_1, \dots, \ell_n, a\}$ is a realization of the configuration \mathcal{C}^{I} , and $f^{\text{I}}(t; \ell_1, \dots, \ell_n, a)$, the joint density of the random variables of interest, is defined in the appendix (see (6.5))

and (6.15)) while $p_0(a)$ is defined in (2.1). In Laplace space with respect to time, (3.3) reads (using the notations defined in (2.3)):

$$\mathcal{L}_t F_n^I(t; \ell) = \hat{F}_n^I(s; \ell) = \left(\int_0^\ell d\tau \rho(\tau) e^{-s\tau} \right)^n \int_0^\ell da p_0(a) e^{-sa}, \quad (3.4)$$

thus

$$\hat{F}^I(s; \ell) = \sum_{n \geq 0} \hat{F}_n^I(s; \ell) = \frac{\int_0^\ell da p_0(a) e^{-sa}}{1 - \int_0^\ell d\tau \rho(\tau) e^{-s\tau}}. \quad (3.5)$$

Normalization of the distribution of $\ell_{\max}^I(t)$ can be checked on this equation by letting ℓ go to infinity. We denote the integrals appearing in the right side of (3.5) by

$$I(s; \ell) = \int_0^\ell da p_0(a) e^{-sa}, \quad J(s; \ell) = \int_0^\ell d\tau \rho(\tau) e^{-s\tau}, \quad (3.6)$$

obeying the relation

$$J(s; \ell) = 1 - p_0(\ell) e^{-s\ell} - sI(s; \ell), \quad (3.7)$$

obtained by an integration by parts. It follows that

$$\hat{F}^I(s; \ell) = \frac{I(s; \ell)}{1 - J(s; \ell)} = \frac{I(s; \ell)}{p_0(\ell) e^{-s\ell} + sI(s; \ell)}, \quad (3.8)$$

and finally, in Laplace space, the complementary distribution function of $\ell_{\max}^I(t)$, namely $1 - F^I(t; \ell) = \text{Prob}(\ell_{\max}(t) > \ell)$, reads

$$\frac{1}{s} - \hat{F}^I(s; \ell) = \frac{1}{s} \frac{p_0(\ell) e^{-s\ell}}{p_0(\ell) e^{-s\ell} + sI(s; \ell)} = \frac{1}{s} \frac{1}{1 + sI(s; \ell) e^{s\ell} / p_0(\ell)}. \quad (3.9)$$

From (3.9) we can easily extract the first moment of $\ell_{\max}^I(t)$ by noting that

$$\langle \ell_{\max}^I(t) \rangle = \int_0^\infty d\ell (1 - F^I(t; \ell)). \quad (3.10)$$

Hence, in Laplace space,

$$\begin{aligned} \mathcal{L}_t \langle \ell_{\max}^I(t) \rangle &= \int_0^\infty d\ell \left(\frac{1}{s} - \hat{F}^I(s; \ell) \right), \\ &= \frac{1}{s} \int_0^\infty d\ell \frac{1}{1 + sI(s; \ell) e^{s\ell} / p_0(\ell)}. \end{aligned} \quad (3.11)$$

Eqs. (3.9) and (3.11) will be analyzed below, according to the nature of the distribution of intervals $\rho(\tau)$.

3.2. Probability of record breaking

The second quantity of interest is the probability that the last interval in the sequence \mathcal{C}^I is the longest one:

$$Q^I(t) = \text{Prob}(\ell_{\max}^I(t) = A_t) = \text{Prob}(A_t > \max(\tau_1, \dots, \tau_N)). \quad (3.12)$$

Its computation is similar to what was done above for $\ell_{\max}^I(t)$. Let

$$Q^I(t) = \sum_{n \geq 0} Q_n^I(t), \quad (3.13)$$

where

$$\begin{aligned} Q_n^I(t) &= \text{Prob}(A_t > \max(\tau_1, \dots, \tau_N), N_t = n), \\ &= \int_0^\infty da p_0(a) \int_0^a d\ell_1 \rho(\ell_1) \dots \int_0^a d\ell_n \rho(\ell_n) \delta\left(\sum_{i=1}^n \ell_i + a - t\right). \end{aligned} \quad (3.14)$$

In Laplace space we have

$$\mathcal{L}_t Q_n^I(t) = \hat{Q}_n^I(s) = \int_0^\infty da p_0(a) e^{-sa} \left(\int_0^a d\tau \rho(\tau) e^{-s\tau} \right)^n, \quad (3.15)$$

and therefore, summing on n ,

$$\hat{Q}^I(s) = \int_0^\infty da \frac{p_0(a) e^{-sa}}{1 - \int_0^a d\tau \rho(\tau) e^{-s\tau}} = \int_0^\infty da \frac{p_0(a) e^{-sa}}{p_0(a) e^{-sa} + sI(s; a)}. \quad (3.16)$$

We note from (3.9) and (3.11) that

$$\hat{Q}^I(s) = s \mathcal{L}_t \langle \ell_{\max}^I(t) \rangle, \quad (3.17)$$

yielding

$$Q^I(t) = \frac{d}{dt} \langle \ell_{\max}^I(t) \rangle, \quad (3.18)$$

since $\langle \ell_{\max}^I(0) \rangle = 0$.

This relationship, which only holds for case I, is actually more general, and applies for any distribution of intervals, i.e., non necessarily generated by a renewal process. Indeed, if t increases by dt , then either $\ell_{\max}^I(t)$ increases by dt , if the last interval, namely A_t , is the longest one, which happens with probability $Q^I(t)$; or stays the same, with probability $1 - Q^I(t)$. Taking the average leads to (3.18) [20].

3.3. Discussion

We now discuss the above results according to the nature of the distribution $\rho(\tau)$ of the intervals.

(i) Exponential distribution of intervals

Let us first consider the special case of an exponential distribution of intervals, $\rho(\tau) = e^{-\tau}$ ($\tau > 0$). This corresponds to the simplest renewal process, where the events form a Poisson process. The starting point of our analysis is (3.9) where the functions $I(s; \ell)$ and $J(s; \ell)$ are given in (3.6), (3.7). In this case, $p_0(a) = e^{-a}$, hence $J(s; \ell) = I(s; \ell)$, and we obtain

$$\frac{1}{s} - \hat{F}^I(s; \ell) = \frac{1+s}{s} \frac{1}{1 + s e^{(1+s)\ell}}. \quad (3.19)$$

Thus from (3.11), one has

$$\mathcal{L}_t \langle \ell_{\max}^I(t) \rangle = \frac{1+s}{s} \int_0^\infty d\ell \frac{1}{1 + s e^{(1+s)\ell}} = \frac{1}{s} \ln \left(1 + \frac{1}{s} \right). \quad (3.20)$$

By inverting the Laplace transform, we obtain

$$\langle \ell_{\max}^I(t) \rangle = E(t) = \int_0^t du \frac{1 - e^{-u}}{u} = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \frac{t^k}{k!}, \quad (3.21)$$

where $E(t)$ is defined by the second equality. At large times we have $E(t) \approx \ln t + \gamma$, where γ is the Euler constant.

The asymptotic distribution of $\ell_{\max}^I(t)$ is actually related to the Gumbel distribution. Indeed from (3.19), we have, for s small and ℓ large,

$$\hat{F}^I(s; \ell) \approx \frac{1}{s + e^{-\ell}}, \quad (3.22)$$

hence

$$F^I(t; \ell) \approx e^{-e^{-\ell}t} = e^{-e^{-(\ell - \ln t)}}. \quad (3.23)$$

In other words, we have, asymptotically,

$$\ell_{\max}^I(t) \approx \ln t + Z^G, \quad (3.24)$$

where Z^G follows the standard Gumbel distribution, with $\langle Z^G \rangle = \gamma$, in agreement with what was found above.

From (3.18) and (3.21) we obtain

$$Q^I(t) = \frac{1 - e^{-t}}{t} \approx \frac{1}{t}. \quad (3.25)$$

The right sides of (3.24) and (3.25) are similar to the forms corresponding to n i.i.d. exponential random variables, if one replaces t by n . This stems from the fact that for an exponential distribution of intervals, A_t has the same distribution as the other intervals, and furthermore the number of intervals $N_t \approx t$, at large times ($\langle \tau \rangle = 1$ here). However, N_t is distributed, and therefore the intervals are not strictly i.i.d. random variables. In summary, in the exponential case, $\ell_{\max}^I(t)$ behaves asymptotically as the largest of $N_t \sim t$ i.i.d random variables, and $Q^I(t)$ scales as $1/\langle N_t \rangle \approx 1/t$.

One can check that, for a Gaussian distribution, $\langle \ell_{\max}^I(t) \rangle \sim (\ln t)^{1/2}$, a result which is asymptotically equivalent to its counterpart for i.i.d. Gaussian variables. Instead, $Q^I(t) \sim 1/(t\sqrt{\ln t})$, obtained by derivation of $\langle \ell_{\max}^I(t) \rangle$ with respect to t , is not asymptotically equivalent to $1/n$. For a uniform distribution of interval, one would find an exponential distribution for the rescaled maximum:

$$\ell_{\max}(t) \approx 1 - \frac{\langle \tau \rangle}{t} Z^{Exp}, \quad (3.26)$$

where Z^{Exp} is exponentially distributed, in line with what is expected from the knowledge of the i.i.d. case. And again $Q^I(t)$ is not equivalent to $Q(n)$. The same treatment can be given for any distribution in the Weibull class.

(ii) *Broad distribution of intervals with index $0 < \theta < 1$*

We now consider intervals with broad distribution $\rho(\tau) \sim \tau^{-1-\theta}$ for large τ (see Eq. 2.2). The starting point of our analysis is again (3.9) with $I(s; \ell)$ and $J(s; \ell)$ given in (3.6), (3.7). We find that in the limit $s \rightarrow 0$, $\ell \rightarrow \infty$, keeping $x = s\ell$ fixed, $I(s; \ell)$ and $J(s; \ell)$ take the scaling forms

$$\begin{aligned} I(s; \ell) &\approx \tau_0^\theta s^{\theta-1} \int_0^{s\ell} du u^{-\theta} e^{-u}, \\ 1 - J(s; \ell) &\approx \tau_0^\theta s^\theta \left((s\ell)^{-\theta} e^{-s\ell} + \int_0^{s\ell} du u^{-\theta} e^{-u} \right). \end{aligned} \quad (3.27)$$

Hence by injecting these scaling forms (3.27) into (3.9) one finds

$$\frac{1}{s} - \hat{F}^I(s; \ell) \approx \frac{1}{s} \hat{f}_V^I(s\ell), \quad (3.28)$$

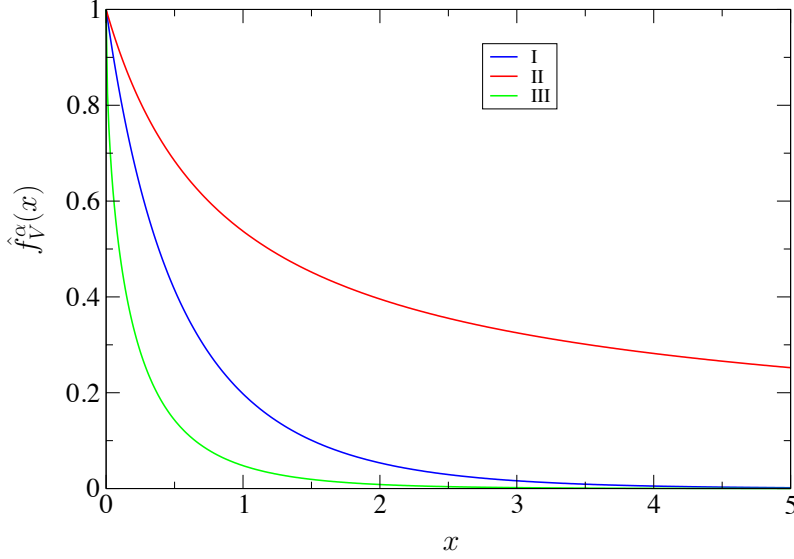


Figure 2. Laplace transforms $\hat{f}_V^\alpha(x) = \langle e^{-xV} \rangle$ of the scaling functions $f_V^\alpha(v)$ for the three cases $\alpha = \text{I, II, III}$ and $\theta = 1/2$ (see (3.29), (4.17) and (5.19)). From top to bottom: II, I, III. The properties of these functions are discussed in the text.

with

$$\hat{f}_V^{\text{I}}(x) = \frac{1}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}}, \quad (3.29)$$

where, as we show below, the scaling function $\hat{f}_V^{\text{I}}(x)$ has a natural probabilistic interpretation. This function can also be written as

$$\hat{f}_V^{\text{I}}(x) = \frac{1}{{}_1F_1(1, 1 - \theta, x)}, \quad (3.30)$$

where ${}_1F_1(1, 1 - \theta, x)$ is a confluent hypergeometric function, simply related to the incomplete gamma function

$$\Gamma(a, x) = \int_x^\infty du u^{a-1} e^{-u}. \quad (3.31)$$

For $\theta = 1/2$, (3.29) reduces to

$$\hat{f}_V^{\text{I}}(x) = \frac{1}{1 + \sqrt{\pi x} e^x \operatorname{erf} \sqrt{x}}, \quad (3.32)$$

which is plotted in figure 2.

From (3.29) we deduce the following. First, using (3.11), analyzed in the small s limit (corresponding to large time), we have

$$\mathcal{L}_t \langle \ell_{\max}^{\text{I}}(t) \rangle \approx \frac{1}{s^2} \int_0^\infty dx \frac{1}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}}, \quad (3.33)$$

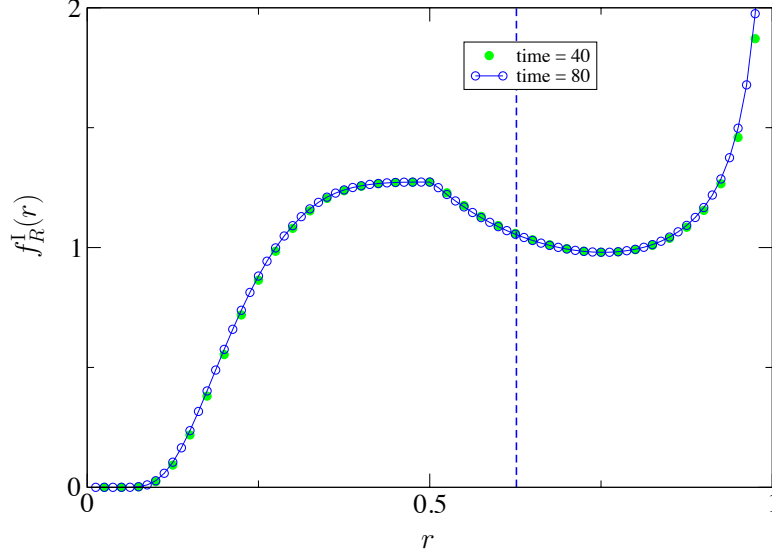


Figure 3. Limiting density $f_R^I(r)$ of the scaling variable $R_t = \ell_{\max}^I(t)/t$ at large times, for $\theta = 1/2$. This was obtained from the analytical expression of the generating function of $\ell_{\max}^I(t)$ for a random walk, after 40 steps (green full circles) and 80 steps (blue empty circles), while the line is a guide to the eyes, connecting the blue circles. The good collapse of the data demonstrates that the scaling is already very good for such values of the number of steps (see text). The vertical dotted line indicates the value of the first moment $\langle R \rangle \approx 0.626\,508\dots$ (see (3.35), (3.43)).

and therefore

$$\lim_{t \rightarrow \infty} \frac{1}{t} \langle \ell_{\max}^I(t) \rangle = Q_{\infty}^I = \int_0^{\infty} dx \hat{f}_V(x) = \int_0^{\infty} dx \frac{1}{1 + x^{\theta} e^x \int_0^x du u^{-\theta} e^{-u}}. \quad (3.34)$$

In particular, for $\theta = 1/2$, (3.34) yields:

$$Q_{\infty}^I = \int_0^{\infty} dx \frac{1}{1 + \sqrt{\pi x} e^x \operatorname{erf} \sqrt{x}} = 0.626\,508\dots, \quad (3.35)$$

recovering a result of Pitman and Yor [34] (see also [35]).

Although we naturally focused above on the variable $\ell_{\max}^I(t)/t$, it turns out that the probability density function of the random variable $t/\ell_{\max}^I(t)$ is simpler to study. Let us denote those two random variables by R_t and $V_t = 1/R_t$:

$$R_t = \frac{\ell_{\max}^I(t)}{t}, \quad V_t = \frac{t}{\ell_{\max}^I(t)}. \quad (3.36)$$

|| Note that (3.34) was used in [36] to demonstrate that $\ell_{\max}^I(t)$ for fractional Brownian motion of Hurst index H (and hence $\theta = 1 - H$ [28]) carries the signature of non-Markovian effects.

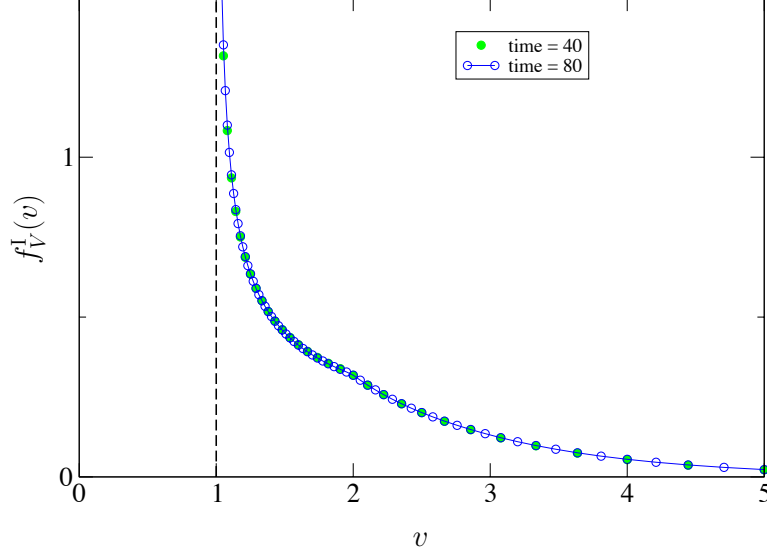


Figure 4. Limiting probability density $f_V^I(v)$, where $V_t = t/\ell_{\max}^I(t)$ as $t \rightarrow \infty$ for $\theta = 1/2$, obtained from the data of figure 3 (using $V_t = 1/R_t$). The tail of this function is exponential, with decay rate approximately equal to 0.854 (see text).

Then, as shown below, (3.29) expresses the fact that in the asymptotic late-time regime the random variables

$$R = \lim_{t \rightarrow \infty} R_t, \quad V = \lim_{t \rightarrow \infty} V_t, \quad (3.37)$$

have time-independent limiting distributions, denoted by $f_R^I(r)$ and $f_V^I(v)$, simply related by

$$f_R^I(r) = \frac{1}{r^2} f_V^I\left(\frac{1}{r}\right). \quad (3.38)$$

It turns out that the scaling function $\hat{f}_V^I(x)$ in (3.29) is precisely the Laplace transform of $f_V^I(v)$ with respect to v , as suggested by the notation. This can be simply demonstrated as follows. At any finite time,

$$1 - F^I(t; \ell) = \text{Prob}(\ell_{\max}^I(t) > \ell) = \text{Prob}(V_t < v \equiv t/\ell). \quad (3.39)$$

In the large time t limit when V_t assumes a stationary distribution $f_V^I(v)$ one has

$$\text{Prob}(V_t < v \equiv t/\ell) \approx \int_0^{t/\ell} dv f_V^I(v). \quad (3.40)$$

Hence, inserting this expression into (3.39) and performing a Laplace transform of both sides of the latter with respect to t , one obtains

$$\frac{1}{s} - \hat{F}^I(s; \ell) \approx \int_0^\infty dt e^{-st} \int_0^{t/\ell} dv f_V^I(v) = \frac{1}{s} \int_0^\infty dv f_V^I(v) e^{-(s\ell)v}, \quad (3.41)$$

where we used an integration by parts to obtain the last equality. Note that this result is actually valid only for small s (since we used (3.40) which itself is valid only for large t). By comparing (3.28) and (3.41) we finally obtain

$$\hat{f}_V^I(x) = \langle e^{-xV} \rangle = \int_0^\infty dv e^{-xv} f_V^I(v). \quad (3.42)$$

Hence (3.29) yields back the result of Lamperti [18].

Using (3.42) we can recast (3.34) as

$$\lim_{t \rightarrow \infty} \frac{1}{t} \langle \ell_{\max}^I(t) \rangle = Q_\infty^I = \int_0^\infty dx \hat{f}_V^I(x) = \left\langle \frac{1}{V} \right\rangle = \langle R \rangle. \quad (3.43)$$

More generally, higher moments of R can be obtained as

$$\langle R^p \rangle = \frac{1}{\Gamma(p)} \int_0^\infty dx x^{p-1} \hat{f}_V^I(x). \quad (3.44)$$

For instance, if $\theta = 1/2$, $\langle R^2 \rangle = 0.4565\dots$, $\langle R^3 \rangle = 0.3653\dots$, which have no explicit expressions. On the other hand the first few moments of V have simple rational expressions (see also [18]):

$$\langle V \rangle = \frac{1}{1-\theta}, \quad \langle V^2 \rangle = \frac{2}{(1-\theta)^2(2-\theta)}, \dots \quad (3.45)$$

As discussed in section 2, for i.i.d. random variables the ratio of the sum to the maximal term has a limiting distribution, $f_W(w)$, as found in [33], which is identical to $f_V^I(v)$. The expression given in [1, 33] reads

$$\hat{f}_W(x) = \frac{e^{-x}}{1 - \theta \int_0^1 du u^{-1-\theta} (e^{-ux} - 1)}, \quad (3.46)$$

where $W = \lim_{t \rightarrow \infty} t_n / \tau_{\max}(n)$ (see (2.15), (2.16)) and x is the Laplace variable conjugate to w . It is easily seen that $\hat{f}_W(x)$ given by (3.46) and $\hat{f}_V^I(x)$ given by (3.29) are identical.

Let us now analyze the generic features of the distributions $f_R^I(r)$ and $f_V^I(v)$. As in other similar problems [18, 19, 23, 24, 25, 26, 27], one expects that $f_V^I(v)$ has a different expression on each interval $[k, k+1]$, with $k = 1, 2, \dots$, with a singularity at each integer $v = 2, 3, \dots$ [18], implying that $f_R^I(r)$ has singularities at $r = 1/2, 1/3, \dots$. In particular, for $1 < v < 2$, one has [18]

$$f_V^I(v) = \frac{\sin \pi \theta}{\pi} (v-1)^{\theta-1}, \quad 1 < v < 2, \quad (3.47)$$

as can be seen by inspection of the large x behaviour of (3.29), $\hat{f}_V^I(x) \approx x^{-\theta} e^{-x} / \Gamma(1-\theta)$.

On the other hand, the large v behaviour of $f_V^I(v)$ is given by the analytic structure of its Laplace transform $\hat{f}_V^I(x)$ (3.29), (3.30) in the complex x -plane. The latter is a meromorphic function, with simple poles located at the zeros $x = s_k$ of the hypergeometric function ${}_1F_1(1, 1-\theta, x)$ (see e.g., [19]). These zeros are such that $s_{\pm k} = -\alpha_k \pm i\beta_k$ (with α_k and β_k real) with a negative real part ($\alpha_k > 0$ for all k) and $0 < \alpha_0 < \alpha_1 < \alpha_2 < \dots$. Furthermore, $s_0 = -\alpha_0$ is the only real zero (i.e., $\beta_0 = 0$). The residues of $\hat{f}_V^I(x)$ at the poles $x = s_k$ are given by $-s_k/\theta$, from which we obtain the large v behaviour of $f_V^I(v)$ as

$$f_V^I(v) \approx \frac{\alpha_0}{\theta} e^{-\alpha_0 v} + \frac{2}{\theta} e^{-\alpha_1 v} (\alpha_1 \cos(\beta_1 v) + \beta_1 \sin(\beta_1 v)) + \mathcal{O}(e^{-\alpha_2 v}). \quad (3.48)$$

Thus the first subleading contribution to the large v behaviour contains an oscillating part. For $\theta = 1/2$, a numerical estimation of the roots s_0 and s_1 yields $\alpha_0 = 0.854032\dots$, $\alpha_1 = 4.24892\dots$ and $\beta_1 = 6.38312\dots$ (see also [19]). From (3.47) and (3.48), together with (3.38), we finally obtain the asymptotic behaviours of $f_R^I(r)$ as

$$f_R^I(r) \approx \begin{cases} (\alpha_0/\theta) r^{-2} e^{-\alpha_0/r}, & r \rightarrow 0 \\ (\sin(\pi\theta)/\pi)(1-r)^{\theta-1}, & r \rightarrow 1. \end{cases} \quad (3.49)$$

In order to have a graphical representation of the densities $f_R^I(r)$ or $f_V^I(v)$ one could either perform the numerical inversion of the Laplace transform of $\hat{f}_V^I(v)$, given by (3.29), or perform a simulation of the process yielding histograms of these densities. Restricting to the case where $\theta = 1/2$, a third method consists in using the expression of the distribution function of $\ell_{\max}^I(t)$, given as an explicit series for the case of the longest lasting record for a random walk [16]. This case is the discrete counterpart (where t is a discrete variable) of the continuous renewal process studied in the present work. In [16] (see (28) therein) we obtained an explicit expression for the generating function $\tilde{F}^I(z; \ell) = \sum_{t \geq 0} F^I(t; \ell) z^t$. Hence the value of $F^I(t; \ell)$ can be read off from the expansion of $\tilde{F}^I(z; \ell)$ close to $z = 0$. This procedure makes sense as (3.29) demonstrates that there is universality of the result in the scaling regime since the scale τ_0 disappears in the expression of the densities in Laplace space. Figures 3 and 4 depict the densities $f_R^I(r)$ and $f_V^I(v)$ obtained that way (the same method is used for cases II and III). The discontinuities of the first derivatives at $r = 1/2$ for $f_R^I(r)$ and $v = 2$ for $f_V^I(v)$ are noticeable.

Let us finally mention that a quantity similar to (albeit different from) $\ell_{\max}^I(t)$ was studied in [25, 26] in the context of charged polymers. In these works, the authors studied the “loops” of the random walks, i.e., segments whose beginning and ending positions coincide, focussing on the largest of such segments irrespectively of their starting points. Translated into the language of, e.g., random walks, the renewal process considered in the present work corresponds to paths that start at the origin but do not necessarily end there. Despite this simplification, our distribution $f_R^I(r)$ shares several features with the distribution of the largest loop studied, mainly numerically, in [25, 26]: it exhibits a non-analytic behaviour for $r = 1/2, 1/3, \dots$, an essential singularity when $r \rightarrow 0$ and a square root divergence when $r \rightarrow 1$ (see (3.49)).

(iii) *Broad distribution with index $1 < \theta < 2$*

We restart from (3.9)

$$\frac{1}{s} - \hat{F}^I(s; \ell) = \frac{1}{s} \frac{1}{1 + sI(s; \ell)e^{s\ell}/p_0(\ell)}, \quad (3.50)$$

that we want to analyze in the late-time regime ($s \rightarrow 0$), where $\ell_{\max}^I(t)$ and ℓ are large. We have $p_0(\ell) \approx (\tau_0/\ell)^\theta$. In order to avoid the divergence of $I(s; \ell)$ at the lower bound, we write

$$I(s; \ell) = \hat{p}_0(s) - \int_\ell^\infty du p_0(u) e^{-su}. \quad (3.51)$$

We have

$$\hat{p}_0(s) = \frac{1 - \hat{\rho}(s)}{s} \approx \langle \tau \rangle - as^{\theta-1}, \quad (3.52)$$

and

$$\int_\ell^\infty du p_0(u) e^{-su} \approx \tau_0^\theta \int_\ell^\infty du u^{-\theta} e^{-su} = \tau_0^\theta s^{\theta-1} \Gamma(1 - \theta, s\ell). \quad (3.53)$$

So,

$$\frac{sI(s; \ell)e^{s\ell}}{p_0(\ell)} \approx \frac{e^{s\ell}s\ell^\theta}{\tau_0^\theta} (\langle \tau \rangle - s^{\theta-1}(a + \tau_0^\theta \Gamma(1-\theta, s\ell))) \quad (3.54)$$

that we have to further analyze.

- (i) The second term in the parentheses $s^{\theta-1}(a + \tau_0^\theta \Gamma(1-\theta, s\ell))$ is subleading compared to the first one, $\langle \tau \rangle$.
- (ii) One has to compare the two products $s\ell$ and $s\ell^\theta$. It is clear that $\ell \sim s^{-1/\theta}$ because then $s\ell \sim s^{1-1/\theta}$ is small, while $\ell \sim s^{-1}$ yields $s\ell^\theta \sim s^{1-\theta}$ which is large.

We thus have $e^{s\ell} \approx 1$. We are left with

$$\frac{1}{s} - \hat{F}^I(s; \ell) \approx \frac{1}{s} \frac{1}{1 + s\langle \tau \rangle (\ell/\tau_0)^\theta}, \quad (3.55)$$

or

$$\hat{F}^I(s; \ell) \approx \frac{1}{s + (\ell/\tau_0)^{-\theta}/\langle \tau \rangle}, \quad (3.56)$$

yielding finally the distribution function of $\ell_{\max}^I(t)$,

$$F^I(t; \ell) \approx e^{-t/\langle \tau \rangle (\ell/\tau_0)^{-\theta}}. \quad (3.57)$$

Setting

$$\ell_{\max}^I(t) = \tau_0 \left(\frac{t}{\langle \tau \rangle} \right)^{1/\theta} Z_t, \quad (3.58)$$

we have, as $t \rightarrow \infty$, $Z_t \rightarrow Z^F$, with limiting distribution

$$\text{Prob}(Z^F < x) = e^{-1/x^\theta} \quad (3.59)$$

which is the Fréchet law. Therefore

$$\langle \ell_{\max}^I(t) \rangle \approx \tau_0 \left(\frac{t}{\langle \tau \rangle} \right)^{1/\theta} \underbrace{\langle Z^F \rangle}_{\Gamma(1-1/\theta)}. \quad (3.60)$$

In other words, $\ell_{\max}^I(t)$, the maximum of $\tau_1, \tau_2, \dots, \tau_N, A_t$ has asymptotically the same distribution as n i.i.d. random variables with common density $\rho(\tau)$. As seen later, the same result also holds for cases II and III. Furthermore, the analysis done above still holds for any $\theta > 1$ because the existence of regular terms of higher order in (3.52) does not affect the analysis of (3.54). Figure 5 depicts the analytical prediction (3.60) for $\theta = 3$, and also compares it to simulations for all three cases.

Finally, taking the temporal derivative of (3.60), we obtain the probability of record breaking

$$Q^I(t) \approx \frac{\Gamma(1-1/\theta)\tau_0}{\theta\langle \tau \rangle} \left(\frac{t}{\langle \tau \rangle} \right)^{-(1-1/\theta)}. \quad (3.61)$$

This probability recast in terms of $n \sim t/\langle \tau \rangle$ reads $Q^I(t) \sim 1/n^{1-1/\theta}$, which differs from the i.i.d. result $Q(n) = 1/n$. The enhancement factor $n^{1/\theta}$ can be traced back to the role of the last interval A_t , as a simple calculation demonstrates. Indeed, considering the random intervals $\tau_1, \tau_2, \dots, \tau_N, A_t$ as independent, with $p_0(a)/\langle \tau \rangle$ for the density of A_t (see [5]), yields indeed (3.61).

To close, let us underline that, when $0 < \theta < 1$, the results found above are universal i.e., only depend on the tail of the distribution $\rho(\tau)$, which is corroborated by the fact that the scale τ_0 disappears of the expressions. This universality does not hold for narrow distributions or for broad distributions with $\theta > 1$.

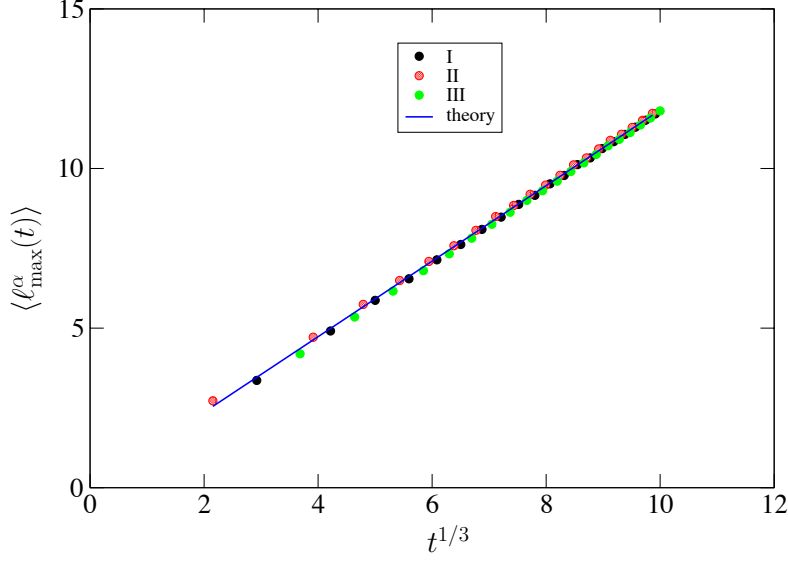


Figure 5. Simulations of $\langle \ell_{\max}^\alpha(t) \rangle$ for the cases $\alpha = \text{I}$ (black circles), $\alpha = \text{II}$ (red circles) and $\alpha = \text{III}$ (green circles) compared to the analytical prediction (3.60) with $\theta = 3$ (solid line).

4. Case II

We now proceed to the sequence of intervals \mathcal{C}^{II} , defined in (2.8), following the line of reasoning employed for case I in the previous section.

4.1. Distribution of $\ell_{\max}^{\text{II}}(t)$

We now consider

$$\ell_{\max}^{\text{II}}(t) = \max(\tau_1, \dots, \tau_{N+1}), \quad (4.1)$$

whose distribution function,

$$F^{\text{II}}(t; \ell) = \text{Prob}(\ell_{\max}^{\text{II}}(t) \leq \ell), \quad (4.2)$$

can be computed by the same method as for case I. We first consider the joint distribution

$$\begin{aligned} F_n^{\text{II}}(t; \ell) &= \text{Prob}(\ell_{\max}^{\text{II}}(t) \leq \ell, N_t = n) \\ &= \int_0^\ell d\ell_1 \rho(\ell_1) \dots \int_0^\ell d\ell_n \rho(\ell_n) \int_0^\ell d\ell_{n+1} \rho(\ell_{n+1}) I\left(\sum_{i=1}^n \ell_i < t < \sum_{i=1}^n \ell_i + \ell_{n+1}\right), \end{aligned} \quad (4.3)$$

where $\{\ell_1, \dots, \ell_n, \ell_{n+1}\}$ is a realization of the configuration \mathcal{C}^{II} , and where $I(\cdot) = 1$ if the condition inside the parentheses is satisfied and $I(\cdot) = 0$ otherwise (see (6.9)). In

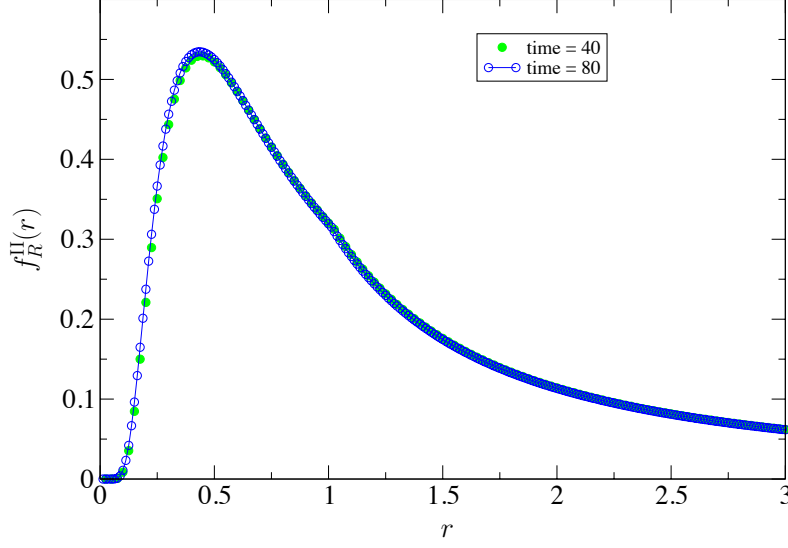


Figure 6. Density $f_R^{\text{II}}(r)$ of the scaling variable R for $\theta = 1/2$, obtained by the same method as in figure 3. The green full circles correspond to a random walk of 40 steps and the blue empty ones to a random walk of 80 steps. The line is a guide to the eyes connecting the blue circles. The tail has exponent $-3/2$, hence the first moment $\langle R \rangle$ of this distribution is infinite. The discontinuity of the derivative at $r = 1$ is clearly visible.

Laplace space we have

$$\begin{aligned} \mathcal{L}_t F_n^{\text{II}}(t; \ell) &= \hat{F}_n^{\text{II}}(s; \ell) \\ &= \left(\int_0^\ell d\tau \rho(\tau) e^{-s\tau} \right)^n \int_0^\ell d\ell_{n+1} \rho(\ell_{n+1}) \frac{1 - e^{-s\ell_{n+1}}}{s}. \end{aligned} \quad (4.4)$$

Thus

$$\hat{F}^{\text{II}}(s; \ell) = \sum_{n \geq 0} \hat{F}_n^{\text{II}}(s; \ell) = \frac{1}{1 - J(s; \ell)} \int_0^\ell d\ell_{n+1} \rho(\ell_{n+1}) \frac{1 - e^{-s\ell_{n+1}}}{s}, \quad (4.5)$$

and

$$\frac{1}{s} - \hat{F}^{\text{II}}(s; \ell) = \frac{1}{s} \frac{p_0(\ell)}{1 - J(s; \ell)} = \frac{1}{s} \frac{e^{s\ell}}{1 + sI(s; \ell)e^{s\ell}/p_0(\ell)}, \quad (4.6)$$

so that

$$\mathcal{L}_t \langle \ell_{\max}^{\text{II}}(t) \rangle = \frac{1}{s} \int_0^\infty d\ell \frac{e^{s\ell}}{1 + sI(s; \ell)e^{s\ell}/p_0(\ell)}. \quad (4.7)$$

Normalization of the distribution of $\ell_{\max}^{\text{II}}(t)$ can be checked on (4.5) by letting $\ell \rightarrow \infty$.

Note that the expression (4.6) is simply related to (3.9). By inversion, we obtain the relation

$$F^{\text{II}}(t; \ell) = F^{\text{I}}(t + \ell, \ell). \quad (4.8)$$

4.2. Probability of record breaking

We now have

$$\begin{aligned} Q^{\text{II}}(t) &= \text{Prob}(\ell_{\text{max}}^{\text{II}}(t) = \tau_{N+1}) = \text{Prob}(\tau_{N+1} > \max(\tau_1, \dots, \tau_N)) \\ &= \sum_{n \geq 0} Q_n(t) = \sum_{n \geq 0} \text{Prob}(\tau_{N+1} > \max(\tau_1, \dots, \tau_n), N_t = n). \end{aligned} \quad (4.9)$$

Explicitly, (setting $\ell_{n+1} = y$, for short),

$$Q_n^{\text{II}}(t) = \int_0^\infty dy \rho(y) \int_0^y d\ell_1 \rho(\ell_1) \dots \int_0^y d\ell_n \rho(\ell_n) I\left(\sum_{i=1}^n \ell_i < t < \sum_{i=1}^n \ell_i + y\right). \quad (4.10)$$

In Laplace space, after summing on n , we obtain

$$\hat{Q}^{\text{II}}(s) = \frac{1}{s} \int_0^\infty dy \frac{\rho(y)(1 - e^{-sy})}{1 - \int_0^y d\ell \rho(\ell) e^{-s\ell}} = \frac{1}{s} \int_0^\infty dy \frac{\rho(y)(1 - e^{-sy})}{p_0(y)e^{-sy} + sI(s; y)}, \quad (4.11)$$

using (3.7). As can be seen on (4.7) and (4.11), there is no simple relationship, as in (3.18), between $\ell_{\text{max}}^{\text{II}}(t)$ and $Q^{\text{II}}(t)$ in the present case.

4.3. Discussion

We now review the behaviours of $\ell_{\text{max}}^{\text{II}}(t)$ and $Q^{\text{II}}(t)$ according to the type of the distribution of intervals $\rho(\tau)$.

(i) Exponential distribution

We have, from (4.6),

$$\frac{1}{s} - \hat{F}^{\text{II}}(s; \ell) = \frac{1+s}{s} \frac{e^{s\ell}}{1 + s e^{(1+s)\ell}}, \quad (4.12)$$

from which no simple explicit expression of $\langle \ell_{\text{max}}^{\text{II}}(t) \rangle$ can be obtained by inversion. Numerically one finds that $\langle \ell_{\text{max}}^{\text{II}}(t) \rangle$ is very close to $\langle \ell_{\text{max}}^{\text{I}}(t) \rangle$, i.e.,

$$\langle \ell_{\text{max}}^{\text{II}}(t) \rangle \approx \ln t + \gamma. \quad (4.13)$$

Moreover, it is easy to show from (4.12) that, for s small and ℓ large, one has

$$\hat{F}^{\text{II}}(s; \ell) \approx \hat{F}^{\text{I}}(s; \ell) \approx \frac{1}{s + e^{-\ell}}, \quad (4.14)$$

which, as in (3.23), shows that $(\ell_{\text{max}}^{\text{II}}(t) - \ln t)$ is distributed according to a Gumbel distribution.

On the other hand we find that

$$\hat{Q}^{\text{II}}(s) = \frac{1+s}{s} \int_0^\infty dy \frac{e^{sy} - 1}{1 + s e^{(1+s)y}}, \quad (4.15)$$

from which, again, no simple explicit expression of $Q^{\text{II}}(t)$ can be obtained. For s small, we have $\hat{Q}^{\text{II}}(s) \approx (1/2)(\ln s)^2$, yielding

$$Q^{\text{II}}(t) \approx \frac{\ln t}{t}, \quad (4.16)$$

which is fully compatible with numerical simulations.

(ii) *Broad distribution with index $0 < \theta < 1$*

It turns out, as explained below, that the first moment of $\ell_{\max}^{\text{II}}(t)$ is not defined. In contrast, $Q^{\text{II}}(t)$ has a well defined limit at large times. This probability was investigated previously by Scheffer [37] as the probability that the last excursion of Brownian motion containing t is the longest ¶.

The aim of this section is to recover the result of Scheffer by simple methods, as already done in [20, 16], and to extend it to any value of $0 < \theta < 1$. We will also determine the density of $\ell_{\max}^{\text{II}}(t)$ in the present case. Using the same methods as for case I, we have, in the scaling regime, starting from (4.6):

$$\frac{1}{s} - \hat{F}^{\text{II}}(s; \ell) \approx \frac{1}{s} \hat{f}_V^{\text{II}}(s\ell), \quad \hat{f}_V^{\text{II}}(x) = \frac{e^x}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}}. \quad (4.17)$$

As for case I, the random variables R_t and V_t (both now defined between 0 and ∞) have limiting distributions in this regime, denoted by $f_R^{\text{II}}(r)$ and $f_V^{\text{II}}(v)$, and the scaling function $\hat{f}_V^{\text{II}}(x)$ (4.17) is the Laplace transform with respect to v of the latter. In the particular case where $\theta = 1/2$,

$$\hat{f}_V^{\text{II}}(x) = \frac{1}{e^{-x} + \sqrt{\pi x} \operatorname{erf} \sqrt{x}} \approx \frac{1}{\sqrt{\pi x}}. \quad (4.18)$$

The integral of this expression diverges, hence $\langle \ell_{\max}^{\text{II}} \rangle / t$ diverges. The explanation of this result is that the largest interval of the sequence \mathcal{C}^{II} has a finite probability of being the last one, τ_{N+1} , which itself does not possess a finite first moment. This also holds for any $0 < \theta < 1$.

Comparing (3.29) and (4.17) we notice that $\hat{f}_V^{\text{II}}(x) = e^x \hat{f}_V^{\text{I}}(x)$, hence we have the relation $f_V^{\text{II}}(v) = f_V^{\text{I}}(v + 1)$ (which can also be deduced directly from (4.8)). This implies the relation $f_R^{\text{II}}(r) = (r + 1)^{-2} f_R^{\text{I}}(r / (r + 1))$. Hence the asymptotic behaviours of $f_V^{\text{II}}(v)$ and $f_R^{\text{II}}(r)$ follow straightforwardly from (3.47), (3.48) and (3.49). Besides, as for case I, the first moments of V have simple explicit expressions (see (3.45)). More generally we have, with p an integer,

$$\langle (V^{\text{II}})^p \rangle = \langle (V^{\text{I}} - 1)^p \rangle. \quad (4.19)$$

On the other hand,

$$Q_\infty^{\text{II}} = \lim_{t \rightarrow \infty} Q^{\text{II}}(t) = \lim_{s \rightarrow 0} s \hat{Q}^{\text{II}}(s) = \theta \int_0^\infty dx \frac{(e^x - 1)/x}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}} \quad (4.20)$$

is finite. For $\theta = 1/2$, one recovers the result of Scheffer [37],

$$Q_\infty^{\text{II}} = \frac{1}{2} \int_0^\infty dx \frac{e^x - 1}{x + \sqrt{\pi x}^{3/2} e^x \operatorname{erf} \sqrt{x}} = 0.800\,310\dots \quad (4.21)$$

(iii) *Broad distribution with index $1 < \theta < 2$*

Performing the same analysis as in section 3.3, we conclude that, for $1 < \theta < 2$, comparing (3.9) and (4.6), we have

$$\frac{1}{s} - \hat{F}^{\text{II}}(s; \ell) \approx \frac{1}{s} - \hat{F}^{\text{I}}(s; \ell) \approx \frac{1}{s} \frac{1}{1 + s \langle \tau \rangle (\ell / \tau_0)^\theta}, \quad (4.22)$$

¶ “Let x_t be the duration of the excursion from 0 straddling t . Then it is asked to compute the probability that this present excursion has a record duration, i.e., that x_t is greater than the maximum of the durations of all previous excursions.” [37]

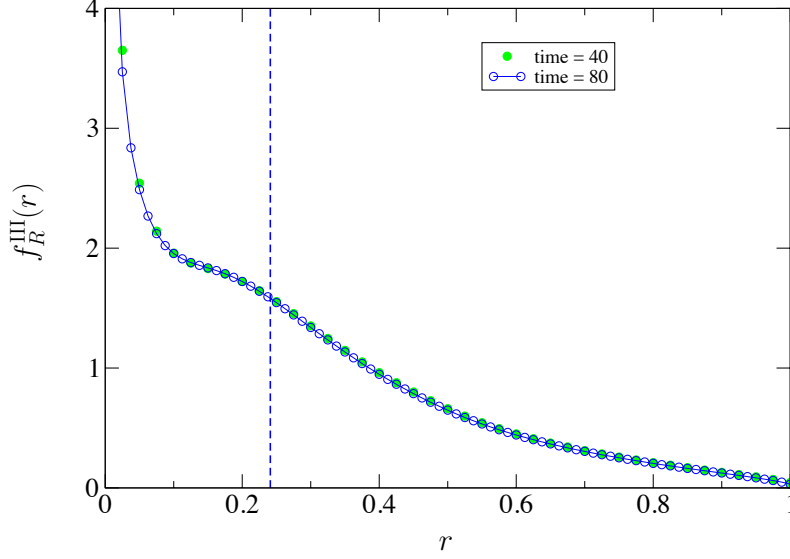


Figure 7. Density $f_R^{\text{III}}(r)$ of the scaling variable R for $\theta = 1/2$, obtained by the same method as in figures 3 and 6. The green full circles correspond to a random walk of 40 steps and the blue empty ones to a random walk of 80 steps. The line is a guide to the eyes connecting the blue circles. The vertical dotted line indicates the value of the first moment $\langle R \rangle \approx 0.241749 \dots$ given in (5.21).

hence the two random variables $\ell_{\max}^{\text{I}}(t)$ and $\ell_{\max}^{\text{II}}(t)$ have asymptotically the same statistics. Restarting from (4.11), we have

$$\hat{Q}^{\text{II}}(s) = \frac{1}{s} \int_0^y dy \frac{\rho(y)(1 - e^{-sy})}{\tau_0^\theta y^{-\theta} e^{-sy} + s\langle\tau\rangle}, \quad (4.23)$$

i.e., using the analysis performed in section 3.3 (see (3.55)),

$$\hat{Q}^{\text{II}}(s) = \theta \int_0^\infty dy \frac{1}{1 + s\langle\tau\rangle(y/\tau_0)^\theta} = \theta \hat{Q}^{\text{I}}(s), \quad (4.24)$$

hence $Q^{\text{II}}(t) = \theta Q^{\text{I}}(t)$, where $Q^{\text{I}}(t)$ is given in (3.61).

5. Case III

We finally consider the sequence of intervals \mathcal{C}^{III} , defined in (2.8), following the line of reasoning employed for the two previous cases, I and II, for the determination of the statistics of the longest interval occurring between 0 and t .

5.1. Distribution of $\ell_{\max}^{\text{III}}(t)$

By definition,

$$\ell_{\max}^{\text{III}}(t) = \max(\tau_1, \dots, \tau_N). \quad (5.1)$$

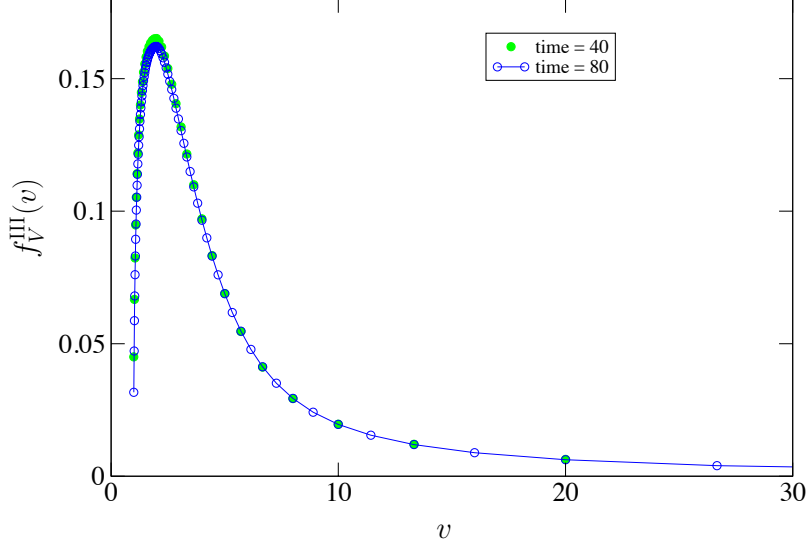


Figure 8. Probability density $f_V^{\text{III}}(v)$, where $V = t/\ell_{\max}^{\text{III}}(t)$ as $t \rightarrow \infty$ for $\theta = 1/2$, obtained from the data of figure 7, using $V_t = 1/R_t$. The function is defined between $v = 1$ and infinity and vanishes as $\sqrt{v-1}$ close to $v = 1$.

We adopt the convention that if there is no event between 0 and t , i.e., $N_t = 0$, then $\ell_{\max}^{\text{III}} = 0$. Its distribution function

$$F^{\text{III}}(t; \ell) = \text{Prob}(\ell_{\max}^{\text{III}}(t) \leq \ell) \quad (5.2)$$

can be computed by the same method as for cases I and II. We have (see (6.13))

$$F_n^{\text{III}}(t; \ell) = \int_0^\ell d\ell_1 \dots \int_0^\ell d\ell_n f^{\text{III}}(t; \ell_1, \dots, \ell_n, n) \quad (5.3)$$

$$= \int_0^\ell d\ell_1 \dots \int_0^\ell d\ell_n \rho(\ell_1) \dots \rho(\ell_n) \int_0^\infty da p_0(a) \delta\left(\sum_{i=1}^n \ell_i + a - t\right), \quad (5.4)$$

where $\{\ell_1, \dots, \ell_n\}$ is a realization of the configuration \mathcal{C}^{III} , yielding, in Laplace space, after summation on n ,

$$\hat{F}^{\text{III}}(s; \ell) = \frac{1 - \hat{\rho}(s)}{s} \frac{1}{1 - J(s; \ell)}, \quad (5.5)$$

hence

$$\frac{1}{s} - \hat{F}^{\text{III}}(s; \ell) = \frac{1}{s} \frac{\hat{\rho}(s) - J(s; \ell)}{1 - J(s; \ell)}. \quad (5.6)$$

Normalization of the distribution of $\ell_{\max}^{\text{III}}(t)$ can be checked on (5.5).

5.2. Probability of record breaking $Q^{\text{III}}(t)$

There are two possible ways of computing this probability. The first one proceeds as for the two previous cases, I and II. We have

$$\begin{aligned} Q^{\text{III}}(t) &= \text{Prob}(\ell_{\max}^{\text{III}}(t) = \tau_N) = \text{Prob}(\tau_N > \max(\tau_1, \dots, \tau_{N-1})), \\ &= \sum_{n \geq 0} Q_n(t) = \sum_{n \geq 0} \text{Prob}(\tau_N > \max(\tau_1, \dots, \tau_{N-1}), N_t = n). \end{aligned} \quad (5.7)$$

Explicitly,

$$Q_n^{\text{III}}(t) = \int_0^\infty d\ell_n \int_0^{\ell_n} d\ell_1 \dots \int_0^{\ell_n} d\ell_{n-1} f^{\text{III}}(t; \ell_1, \dots, \ell_{n-1}, \ell_n, n). \quad (5.8)$$

In Laplace space, after summing on n , we obtain

$$\begin{aligned} \hat{Q}^{\text{III}}(s) &= \hat{p}_0(s) \int_0^\infty dy \frac{\rho(y) e^{-sy}}{1 - \int_0^y d\ell \rho(\ell) e^{-s\ell}}, \\ &= \hat{p}_0(s) \int_0^\infty dy \frac{dJ(s; y)/dy}{1 - J(s; y)}, \end{aligned} \quad (5.9)$$

where $J(s; y) = \int_0^y d\ell \rho(\ell) e^{-s\ell}$. Finally,

$$\hat{Q}^{\text{III}}(s) = \hat{p}_0(s) \int_0^{\hat{\rho}(s)} \frac{dJ}{1 - J} = -\frac{1 - \hat{\rho}(s)}{s} \ln(1 - \hat{\rho}(s)). \quad (5.10)$$

We now show that the same result can be recovered by a more intuitive method, which relies on the idea that the N_t intervals τ_1, \dots, τ_N are expected to play the same role. One is therefore tempted to apply the well known fact, valid for i.i.d. random variables, that the probability for the last variable to be the largest, that is to say, the probability of record breaking, is equal to the inverse number of random variables. We are thus led to write

$$Q_n^{\text{III}}(t) = \text{Prob}(\tau_n > \max(\tau_1, \dots, \tau_{n-1})) = \frac{p_n}{n}, \quad (n > 0), \quad (5.11)$$

where $p_n = \text{Prob}(N_t = n)$, thus

$$Q^{\text{III}}(t) = \text{Prob}(\tau_N > \max(\tau_1, \dots, \tau_{N-1}), N_t = n) = \sum_{n \geq 1} Q_n^{\text{III}}(t) = \left\langle \frac{1}{N_t} \right\rangle, \quad (5.12)$$

(without $N_t = 0$). In Laplace space,

$$\begin{aligned} \hat{Q}^{\text{III}}(s) &= \sum_{n \geq 1} \frac{\hat{p}_n(s)}{n} = \frac{1 - \hat{\rho}(s)}{s} \sum_{n \geq 1} \frac{\hat{\rho}(s)^n}{n}, \\ &= -\frac{1 - \hat{\rho}(s)}{s} \ln(1 - \hat{\rho}(s)), \end{aligned} \quad (5.13)$$

which is (5.10) above.

5.3. Discussion

(i) Exponential distribution of intervals

Using (5.6) and (5.10), we find

$$\mathcal{L}_t \langle \ell_{\max}^{\text{III}}(t) \rangle = \frac{1}{s} \int_0^\infty d\ell \frac{1}{1 + s e^{(s+1)\ell}} = \frac{1}{s(1+s)} \ln \left(1 + \frac{1}{s} \right), \quad (5.14)$$

and

$$\hat{Q}^{\text{III}}(s) = \frac{1}{1+s} \ln \left(1 + \frac{1}{s} \right). \quad (5.15)$$

From these two expressions and from (3.20) it follows that

$$\begin{aligned} \langle \ell_{\max}^{\text{III}}(t) \rangle &= \langle \ell_{\max}^{\text{I}}(t) \rangle - Q^{\text{III}}(t), \\ &\approx \ln t + \gamma, \end{aligned} \quad (5.16)$$

where

$$Q^{\text{III}}(t) = -e^{-t} \int_0^t du \frac{1 - e^u}{u} \approx \frac{1}{t}. \quad (5.17)$$

In practice $\langle \ell_{\max}^{\text{I}}(t) \rangle$ and $\langle \ell_{\max}^{\text{III}}(t) \rangle$, as well as $Q^{\text{I}}(t)$ and $Q^{\text{III}}(t)$, are numerically rapidly indistinguishable in this exponential case.

One can also check from (5.6) that

$$\hat{F}^{\text{III}}(s; \ell) \approx \hat{F}^{\text{I}}(s; \ell) \approx \frac{1}{s + e^{-\ell}}, \quad (5.18)$$

implying, as in (3.23), that $(\ell_{\max}^{\text{III}}(t) - \ln t)$ is distributed according to a Gumbel distribution.

(ii) *Broad distribution of intervals with index $0 < \theta < 1$*

The starting point of our analysis is (5.6). By inserting the small s behaviour (2.4) of $\hat{\rho}(s)$ and (3.27) of $J(s; l)$ into (5.6), one obtains, in the limit $s \rightarrow 0$, $\ell \rightarrow \infty$, keeping $x = s\ell$ fixed:

$$\frac{1}{s} - \hat{F}^{\text{III}}(s; \ell) \approx \frac{1}{s} \hat{f}_V^{\text{III}}(s\ell), \quad \hat{f}_V^{\text{III}}(x) = \frac{1 - x^\theta e^x \int_x^\infty du u^{-\theta} e^{-u}}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}}. \quad (5.19)$$

As for cases I and II, the scaling function $\hat{f}_V^{\text{III}}(x)$ is the Laplace transform with respect to v of the limiting distribution $f_V^{\text{III}}(v)$ of the rescaled variable $V_t = t/\ell_{\max}^{\text{III}}(t)$ (see (3.42)). Besides, from (5.19) we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \langle \ell_{\max}^{\text{III}}(t) \rangle = \int_0^\infty dx \hat{f}_V^{\text{III}}(x) = \int_0^\infty dx \frac{1 - x^\theta e^x \int_x^\infty du u^{-\theta} e^{-u}}{1 + x^\theta e^x \int_0^x du u^{-\theta} e^{-u}}. \quad (5.20)$$

In particular, for $\theta = 1/2$ this yields

$$\lim_{t \rightarrow \infty} \frac{1}{t} \langle \ell_{\max}^{\text{III}}(t) \rangle = \int_0^\infty dx \frac{1 - \sqrt{\pi x} e^x \operatorname{erfc} \sqrt{x}}{1 + \sqrt{\pi x} e^x \operatorname{erf} \sqrt{x}} = 0.241\,749\dots \quad (5.21)$$

Let us briefly discuss the behaviours of the densities $f_R^{\text{III}}(r)$ and $f_V^{\text{III}}(v)$. As for case I (see, e.g., (3.47)), one expects that $f_V^{\text{III}}(v)$ has different expressions on each interval $[k, k+1]$ where $k = 1, 2, \dots$, with singularities at $v = 2, 3, \dots$. On the other hand, the asymptotic behaviour of $f_V^{\text{III}}(v)$ can be easily obtained from its Laplace transform $\hat{f}_V^{\text{III}}(x)$. In particular, for large x , one has $\hat{f}_V^{\text{III}}(x) \sim (\theta/\Gamma(1-\theta))x^{-1-\theta}e^{-x}$ from which it follows that

$$f_V^{\text{III}}(v) \approx \frac{\sin \pi \theta}{\pi} (v-1)^\theta, \quad v \rightarrow 1. \quad (5.22)$$

The large v behaviour of $f_V^{\text{III}}(v)$ is dominated by the non-analytic behaviour of $\hat{f}_V^{\text{III}}(x)$ close to the origin, while the pole of $\hat{f}_V^{\text{III}}(x)$ yields here subleading corrections. One has indeed $\hat{f}_V^{\text{III}}(x) \approx 1 - x^\theta \Gamma(1-\theta)$, when $x \rightarrow 0$, implying

$$f_V^{\text{III}}(v) \approx \theta v^{-1-\theta}, \quad v \rightarrow \infty, \quad (5.23)$$

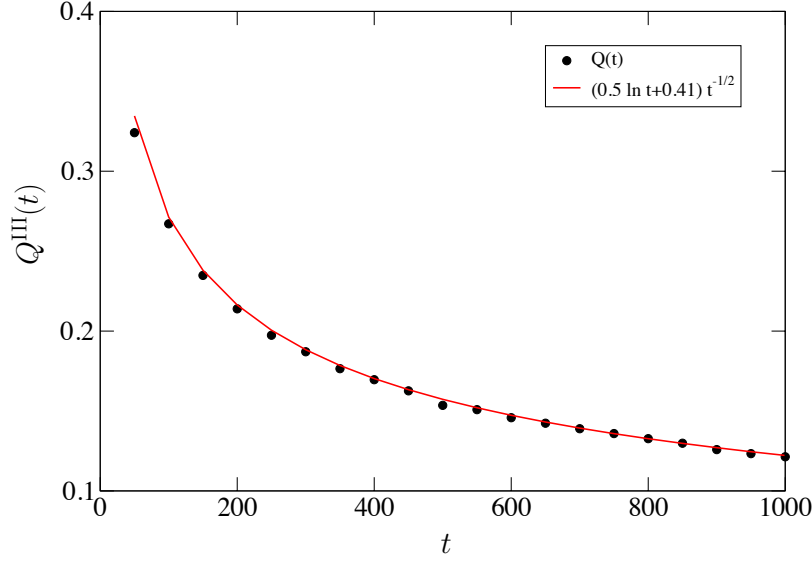


Figure 9. Comparison of the analytical prediction (5.26) (red solid line) for $Q^{\text{III}}(t)$ with simulations (black full circles) with $\theta = 1/2$.

which shows that V has no finite moments. From the asymptotic behaviours of $f_V^{\text{III}}(v)$ in (5.22) and (5.23) one obtains those of $f_R^{\text{III}}(r)$ as

$$f_R^{\text{III}}(r) \approx \begin{cases} \theta r^{\theta-1}, & r \rightarrow 0 \\ (\sin(\pi\theta)/\pi)(1-r)^\theta, & r \rightarrow 1. \end{cases} \quad (5.24)$$

Figure 7 depicts the density $f_R^{\text{III}}(r)$ of the scaling variable R , for $\theta = 1/2$, using the same method as in figures 3 and 6. Figure 8 is deduced from figure 7.

Next, we have

$$\hat{Q}^{\text{III}}(s) \approx -as^{\theta-1} \ln(as^\theta), \quad (5.25)$$

hence

$$Q^{\text{III}}(t) \approx (A \ln t + B) t^{-\theta}, \quad (5.26)$$

with $A = \theta\tau_0^\theta$, and B is a constant depending on τ_0 and θ . For instance, taking a distribution of intervals $\rho(\tau)$ generated by $U^{-1/\theta}$ where U is uniform between 0 and 1, then $\tau_0 = 1$, and if for instance $\theta = 1/2$, we have $B = \gamma/2 + \ln(2/\pi)$ (see figure 9).

(iii) *Broad distribution of intervals with index $1 < \theta < 2$*

Again we find

$$\frac{1}{s} - \hat{F}^{\text{III}}(s; \ell) \approx \frac{1}{s} - \hat{F}^{\text{I}}(s; \ell), \quad (5.27)$$

hence the distribution of $\ell_{\text{max}}^{\text{III}}(t)$ is asymptotically identical to that of $\ell_{\text{max}}^{\text{I}}(t)$.

Finally, a simple analysis yields

$$Q^{\text{III}}(t) = \left\langle \frac{1}{N_t} \right\rangle \approx \frac{\langle \tau \rangle}{t}. \quad (5.28)$$

This behaviour is akin to the case of i.i.d. random variables where $Q(n) = 1/n$, since here we have $\langle N_t \rangle \approx t/\langle \tau \rangle$.

6. Conclusion and discussion

In this paper, we have investigated the statistics of the longest interval in renewal processes. Let us put our results in the context of recent works starting with [13], where the authors studied, among others, the longest lasting record in symmetric random walks. Thanks to the discrete renewal properties of the records of random walks [1, 13], the longest lasting record corresponds precisely to the longest interval in a renewal process with a distribution of intervals $\rho(\tau) \sim \tau^{-1-\theta}$, where $\theta = 1/2$. Ref. [13] considered the situation referred to as case I in the present work (see (2.8)) and obtained the rescaled first moment $\langle \ell_{\text{max}}^{\text{I}}(t) \rangle/t$. A subsequent paper focused on the related question of $\langle \ell_{\text{max}}^{\text{I}}(t) \rangle/t$ for a broader class of continuous renewal processes, where $\rho(\tau) \sim \tau^{-1-\theta}$, with $0 < \theta < 2$, and discussed the relevance of this quantity for stochastic processes in nonequilibrium systems [20]. This reference also studied the probability of record breaking $Q^{\text{I}}(t)$ and discussed briefly the possible extensions of these quantities, $\ell_{\text{max}}^{\text{I}}(t)$ and $Q^{\text{I}}(t)$, to different ensembles denoted by \mathcal{C}^{II} and \mathcal{C}^{III} in the present paper (see (2.8)). Then, coming back to the record statistics of random walks, a detailed analysis of the statistics of $\ell_{\text{max}}^{\alpha}(t)$ and $Q^{\alpha}(t)$ for the three cases $\alpha = \text{I, II, III}$ was recently performed [16].

The present study of the statistics of $\ell_{\text{max}}^{\alpha}(t)$ and of $Q^{\alpha}(t)$ for a large panel of distributions of intervals $\rho(\tau)$ completes the previous aforementioned works. We found a rich variety of behaviours, depending on the tail of $\rho(\tau)$, which we briefly summarize here (see also tables 1, 2 and 3). Concerning the statistics of $\ell_{\text{max}}^{\alpha}(t)$, we showed that, if $\rho(\tau)$ decays faster than $1/\tau^2$, the fluctuations of $\ell_{\text{max}}^{\alpha}(t)$ are given in the large t limit by the classical theory of extreme value statistics for i.i.d. random variables. Indeed the distribution of $\ell_{\text{max}}^{\alpha}(t)$, properly shifted and scaled, converges in the large t limit to one of the standard distributions of extreme value statistics for i.i.d. random variables, namely Gumbel, Fréchet or Weibull, depending on the tail of $\rho(\tau)$, and provided that $\tau^2 \rho(\tau) \rightarrow 0$ when $\tau \rightarrow \infty$. This shows in particular that the global constraint imposing that the sum of the time intervals is fixed to t becomes irrelevant, in this case, for large t . This is however not the case if $\rho(\tau)$ has a power-law decay with tail exponent $0 < \theta < 1$, where the limiting distribution of the rescaled variable $R_t = \ell_{\text{max}}^{\alpha}(t)/t$, computed in the various cases $\alpha = \text{I, II and III}$, is non-trivial. A generic feature of these limiting distributions $f_R^{\alpha}(r)$ is that they exhibit non-analyticities at values $r = 1/k$ with $k = 2, 3, \dots$ for the cases $\alpha = \text{I, III}$ and $k = 1, 2, \dots$ for $\alpha = \text{II}$. Such non-analytic densities have been obtained previously in several related, though different, situations [18, 19, 23, 24, 25, 26, 27]. We refer the reader to figures 3, 6 and 7 for a plot of these densities in the case $\theta = 1/2$, corresponding to the renewal sequence of zeros or records of random walks. Although the variable $R_t = \ell_{\text{max}}^{\alpha}(t)/t$ seems physically more natural, it turns out that the distribution of $V_t = 1/R_t = t/\ell_{\text{max}}^{\alpha}(t)$ is easier to study. In case I, the limiting distribution of V_t had been computed before by Lamperti in [18]. In this case, our results provide a simple alternative derivation of his result. As mentioned in section 2, $f_V^{\text{I}}(v)$ is identical to the distribution found by

Darling for the ratio between the maximum term and the sum of heavy tailed random variables with $0 < \theta < 1$ [34]. We refer the reader to figures 4 and 8 for plots of the limiting distributions $f_V^I(v)$ and $f_V^{III}(v)$ for $\theta = 1/2$ (while $f_V^{II}(v) = f_V^I(v + 1)$). A nice feature of these distributions in the case $0 < \theta < 1$ is that they are completely universal, depending only on the exponent θ and not on the “microscopic” details of the distribution $\rho(\tau)$.

We also found a rich behaviour of the probability of record breaking $Q^\alpha(t)$ depending on the time distribution $\rho(\tau)$. In particular, we showed that in most cases $Q^\alpha(t)$ behaves quite differently for renewal processes and for i.i.d. random variables. Again, the case $\theta < 1$ is particularly interesting as it gives rise to universal constants, for cases I and II. In particular, we recovered in a simple way a result obtained by Scheffer [37] for Q_∞^{II} with $\theta = 1/2$, which we have generalized to any value of θ (see (4.20)).

It is remarkable that the observables $\ell_{\max}^\alpha(t)$ and $Q^\alpha(t)$ are quite sensitive to the last excursion, as the cases I, II and III show different behaviours (see tables 1, 2 and 3). And it will be interesting to extend this study to other observables, like for instance the smallest interval in renewal processes, whose average values were computed in some cases in the context of record statistics [13, 15, 16].

The influence of the last interval was also recently pointed out in the context of anomalous diffusion [38, 39]⁺.

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Appendix

6.1. Notations

Our notations are those commonly used in probability theory: if X is a random variable, then its distribution function reads

$$F_X(x) = \text{Prob}(X < x), \quad (6.1)$$

and its density is

$$f_X(x) = \frac{dF_X(x)}{dx}. \quad (6.2)$$

Likewise, for several random variables, we have

$$F_{X_1, X_2, \dots}(x_1, x_2, \dots) = \text{Prob}(X_1 < x_1, X_2 < x_2, \dots), \quad (6.3)$$

with the associated density $f_{X_1, X_2, \dots}(x_1, x_2, \dots)$. When permitted by the context, we will omit the variables in subscript.

⁺ We are indebted to Eli Barkai for pointing out refs. [38, 39] to us.

6.2. Joint probability densities

(I) The joint probability density of $\tau_1, \dots, \tau_N, A_t, N_t$ is

$$\begin{aligned} f_{\tau_1, \dots, \tau_N, A_t, N_t}(t; \ell_1, \dots, \ell_n, a, n) \\ = \langle \delta(\tau_1 - \ell_1) \dots \delta(\tau_n - \ell_n) \delta(A_t - a) I(t_n < t < t_{n+1}) \rangle, \end{aligned} \quad (6.4)$$

where $I(\cdot) = 1$ if the condition inside the parentheses is satisfied and $I(\cdot) = 0$ otherwise. This yields

$$\begin{aligned} f_{\tau_1, \dots, \tau_N, A_t, N_t}(t; \ell_1, \dots, \ell_n, a, n) \\ = \rho(\ell_1) \dots \rho(\ell_n) p_0(a) \delta\left(\sum_{i=1}^n \ell_i + a - t\right). \end{aligned} \quad (6.5)$$

Its Laplace transform with respect to time reads

$$\begin{aligned} \mathcal{L}_t f_{\tau_1, \dots, \tau_N, A_t, N_t}(t; \ell_1, \dots, \ell_n, a, n) &= \hat{f}_{\tau_1, \dots, \tau_N, A_t, N_t}(s; \ell_1, \dots, \ell_n, a, n) \\ &= \rho(\ell_1) \dots \rho(\ell_n) e^{-s \sum_i \ell_i} p_0(a) e^{-sa}. \end{aligned} \quad (6.6)$$

Laplace transforming with respect to age yields

$$\hat{f}_{\tau_1, \dots, \tau_N, A_t, N_t}(s; \ell_1, \dots, \ell_n, u, n) = \rho(\ell_1) \dots \rho(\ell_n) e^{-s \sum_i \ell_i} \frac{1 - \hat{\rho}(s + u)}{s + u}. \quad (6.7)$$

(II) Likewise, the joint probability density of $\tau_1, \dots, \tau_{N+1}, N_t$ is

$$\begin{aligned} f_{\tau_1, \dots, \tau_{N+1}, N_t}(t; \ell_1, \dots, \ell_{n+1}, n) \\ = \langle \delta(\tau_1 - \ell_1) \dots \delta(\tau_{n+1} - \ell_{n+1}) I(t_n < t < t_{n+1}) \rangle, \end{aligned} \quad (6.8)$$

yielding

$$f_{\tau_1, \dots, \tau_{N+1}, N_t}(t; \ell_1, \dots, \ell_{n+1}, n) = \rho(\ell_1) \dots \rho(\ell_{n+1}) I(t_n < t < t_n + \ell_{n+1}). \quad (6.9)$$

In Laplace space with respect to t , we have

$$\begin{aligned} \mathcal{L}_t f_{\tau_1, \dots, \tau_{N+1}, N_t}(t; \ell_1, \dots, \ell_{n+1}, n) &= \hat{f}_{\tau_1, \dots, \tau_{N+1}, N_t}(s; \ell_1, \dots, \ell_n, \ell_{n+1}, n) \\ &= \rho(\ell_1) \dots \rho(\ell_n) e^{-s \sum_i \ell_i} \rho(\ell_{n+1}) \frac{1 - e^{-s \ell_{n+1}}}{s}. \end{aligned} \quad (6.10)$$

Laplace transforming with respect to ℓ_{n+1} then gives

$$\hat{f}_{\tau_1, \dots, \tau_{N+1}, N_t}(s; \ell_1, \dots, \ell_n, u, n) = \rho(\ell_1) \dots \rho(\ell_n) e^{-s \sum_i \ell_i} \frac{\hat{\rho}(u) - \hat{\rho}(u + s)}{s}. \quad (6.11)$$

(III) Finally, for the third sequence,

$$f_{\tau_1, \dots, \tau_N, N_t}(t; \ell_1, \dots, \ell_n, n) = \langle \delta(\tau_1 - \ell_1) \dots \delta(\tau_n - \ell_n) I(t_n < t < t_{n+1}) \rangle, \quad (6.12)$$

yielding

$$f_{\tau_1, \dots, \tau_N, N_t}(t; \ell_1, \dots, \ell_n, n) = \rho(\ell_1) \dots \rho(\ell_n) \int_0^\infty da p_0(a) \delta\left(\sum_{i=1}^n \ell_i + a - t\right), \quad (6.13)$$

which can alternatively be obtained from (6.5) or (6.9). In Laplace space

$$\mathcal{L}_t f_{\tau_1, \dots, \tau_N, N_t}(t; \ell_1, \dots, \ell_n, n) = \rho(\ell_1) \dots \rho(\ell_n) e^{-s \sum_i \ell_i} \frac{1 - \hat{\rho}(s)}{s}, \quad (6.14)$$

which can consistently be derived from (6.7) and (6.11), setting $u = 0$ in these expressions.

Eqs. (6.5), (6.9) and (6.13) are the building blocks for the analysis performed in the bulk of the text. For short, we denote the joint probability densities associated to the different sequences \mathcal{C}^α by $f^\alpha(\dots)$:

$$\begin{aligned} f^{\text{I}}(t; \ell_1, \dots, \ell_n, a, n) &= f_{\tau_1, \dots, \tau_N, A_t, N_t}(t; \ell_1, \dots, \ell_n, a, n), \\ f^{\text{II}}(t; \ell_1, \dots, \ell_{n+1}, n) &= f_{\tau_1, \dots, \tau_{N+1}, N_t}(t; \ell_1, \dots, \ell_{n+1}, n), \\ f^{\text{III}}(t; \ell_1, \dots, \ell_n, n) &= f_{\tau_1, \dots, \tau_N, N_t}(t; \ell_1, \dots, \ell_n, n). \end{aligned} \quad (6.15)$$

References

- [1] Feller W 1968 1971 *An Introduction to Probability Theory and its Applications* Volumes 1&2 (New York: Wiley)
- [2] Cox D R 1962 *Renewal theory* (London: Methuen)
- [3] Dynkin E B 1955 *Izv. Akad. Nauk. SSSR Ser. Math.* **19** 247
- [4] Dynkin E B 1961 *Selected Translations Math. Stat. Prob.* **1** 171
- [5] Baldassarri A, Bouchaud J P, Dornic I and Godrèche C 1999 *Phys. Rev. E* **59** R20
- [6] Godrèche C and Luck J M 2001 *J. Stat. Phys.* **104** 489
- [7] Lamperti J 1958 *Trans. Amer. Math. Soc.* **88** 380
- [8] Brokmann X, Hermier J P, Messin G, Desbiolles P, Bouchaud J P and Dahan M 2003 *Phys. Rev. Lett.* **90** 120601; Margolin G and Barkai E 2004 *J. Chem. Phys.* **121** 1566; Stefani F D, Hoogenboom J P, and Barkai E 2009 *Phys. Today* **62** 34.
- [9] Majumdar S N, Bray A J, Cornell S J and Sire C 1996 *Phys. Rev. Lett.* **77** 2867
- [10] Derrida B, Hakim V and Zeitak R 1996 *Phys. Rev. Lett.* **77** 2871
- [11] Dornic I and Godrèche C 1998 *J. Phys. A* **31** 5413
- [12] Newman T J and Toroczkai Z 1998 *Phys. Rev. E* **58** R2685
- [13] Dhar A and Majumdar S N 1999 *Phys. Rev. E* **59** 6413
- [14] De Smedt G, Godrèche C and Luck J M 2001 *J. Phys. A* **34** 1247
- [15] Majumdar S N and Ziff R M 2008 *Phys. Rev. Lett.* **101** 050601
- [16] Wergen G, Majumdar S N and Schehr G 2012 *Phys. Rev. E* **6** 011119
- [17] Majumdar S N, Schehr G and Wergen G 2012 *J. Phys. A* **45** 355002
- [18] Godrèche C, Majumdar S N and Schehr G 2014 *J. Phys. A* **47** 255001
- [19] Schehr G and Majumdar S N 2014 in *First-Passage Phenomena and Their Applications*, Eds. Metzler R, Oshanin G and Redner S (Singapore: World Scientific)
- [20] Lamperti J P 1961 *Am. Math. Soc.* **12**(5) 724
- [21] Wendel J G 1964 *Math. Scand.* **14** 21
- [22] Godrèche C, Majumdar S N and Schehr G 2009 *Phys. Rev. Lett.* **102** 240602
- [23] Gnedenko B 1943 *Ann. Math.* **44** 423
- [24] Gumbel E J 1958 *Statistics of Extremes* (New York: Dover)
- [25] Derrida B and Flyvbjerg H 1987 *J. Phys A: Math. Gen.* **20** 5273
- [26] Frachebourg L, Ispolatov I and Krapivsky P L 1995 *Phys. Rev. E* **52** R5727
- [27] Kantor Y and Ertas D 1994 *J. Phys A* **27** L907
- [28] Ertas D and Kantor Y 1997 *Phys. Rev. E* **55** 261
- [29] Godrèche C and Luck J M 2008 *J. Stat. Mech.* P11006
- [30] For a review on persistence: Bray A J, Majumdar S N and Schehr G 2013 *Adv. Phys.* **62** 225
- [31] Derrida B, Bray A J and Godrèche C 1994 *J. Phys. A* **27** L357
- [32] Bray A J, Derrida B and Godrèche C 1994 *Europhys. Lett.* **27** 175
- [33] Renyi A 1962 *Ann. Fac. Sc. Univ. Clermont-Ferrand* **8** 7
- [34] Lévy P 1935 *J. Math.* **14** 347
- [35] Darling D A 1952 *Trans. Amer. Math. Soc.* **73** 95
- [36] Pitman J and Yor M 1997 *Ann. Probab.* **25** 855
- [37] Finch S R 2008 <http://www.people.fas.harvard.edu/~sfinch/>
- [38] Finch S R 2003 *Mathematical constants* (Cambridge: Cambridge University Press) 284
- [39] Garcia-Garcia R, Rosso A and Schehr G 2010 *Phys. Rev. E* **81** 010102(R)
- [40] Scheffer C L 1995 *Stoch. Proc. Appl.* **55** 101
- [41] Barkai E, Aghion E and Kessler D A 2014 *Phys. Rev. X* **4** 021036
- [42] Froemberg D, Schmiedeberg M, Barkai E and Zaburdaev V 2014 preprint arXiv:1412.0984